Bijective Counting

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December 28, 2023

1 Introduction

2 Quick Review on Elementary Counting

Example 2.1. Let n and $r \leq n$ be positive integers. From a committee of n people, we want to select r people to form a subcommittee. In how many ways can we create the subcommittee?

Example 2.2. How many 12-letter words can you make using 3 letter A's, 4 letter B's and 5 letter C's?

3 Bijection with Words

Example 3.3. The roadmap of Townsville is given. Josh wants to go from A to B by going only northwards or eastwards. In how many ways can Josh go?

Example 3.4. Eight targets are hung from the ceiling in the strings of 3, 2 and 3 respectively (see figure XXX). A musketeer wants to shoot down all 8 targets one by one, in a sequence. But they channot shoot a target unless all the targets below it were already shot. Find the number of ways to shoot all 8 targets.

Example 3.5. Find the number of non-negative integer solutions to the equation $x + y + z = 10$.

4 Proving Combinatorial Identities

Example 4.6. Let n be a positive integer. Prove that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

Example 4.7. Let $n \geq 100$ be a positive integer. Simplify: $\binom{n+1}{100} - \binom{n}{99}$.

Example 4.8. Let n be a positive integer. Find a formula for the sum: $1^2 + 2^2 + 3^2 + \cdots + n^2$.

5 Exercises

အောက်ပါပုစ္ဆာတွေမှာ ထူးထူးထွေထွေပြောထားတာမရှိရင် n သည်အမြဲတစေ positive integer တစ်လုံးဖြစ်ပါတယ်။

1. In this exercise, $k \leq n$ is always a positive integer. Count the following objects:
   (a) Ways to choose 5 eggs out of $n \geq 5$ for boiling,
   (b) Quadrilaterals drawn using a set of $n \geq 4$ non-collinear points as vertices,
   (c) Ways to make an k-person ($k \geq 5$) football team from a class of $n \geq 11$ students where 5 particular students must be chosen,
   (d) Ways to create a 4 person team with 2 boys and 2 girls from a class with n boys and $2n$ girls,
   (e) n-letter words with k letter A's and $n-k$ letter B's,
   (f) Ways to select $n \leq 100$ students from a class 100, and then pick k secetaries from the chosen students,
   (g) Ways to choose k vertices of a regular n-gon and colour each of them with either red, green or blue,
   (h) $2n$-letter words with k letter A's, k letter B's and other letters being C or D.
2. Find the number of ways to tile a 1×20 rectangle using 4 identical 1×3 rectangles and 8 identical 1×1 rectangles.
3. A 5 digit-number $\overline{abcde}$ is called increasing if $a < b < c < d < e$. How many increasing numbers are there?
4. Frodo the frog sits at the origin of the xy-plane. When the owner claps, Frodo jumps 1 unit in positive x-direction. When the owner snaps, Frodo jumps 1 unit in positive y-direction.
   (a) In how many ways can Frodo go to the point (n, n)?
   (b) In how many ways can Frodo go onto the line $x + y = n$?
5. A zookeeper has 10 identical bananas to feed his 4 monkeys. In how many ways can he feed? What if every monkey must get at least 2 bananas?
6. Let (w, x, y, z) be integers such that $w + x + y + z = 12$.
   (a) Find the number of such quadruplets with $w, x, y, z \geq 0$.
   (b) Find the number of such quadruplets with $w, x, y, z \geq 1$.

(c) $*$ Find the number of such quadruplets with $w, x, y, z \le 5$.

7. Ten identical red lanterns and five distinct blue lanterns are to be hung along a horizontal string. How many ways are there to hang them if there must be at least 1 red lantern between any two blue lanterns?

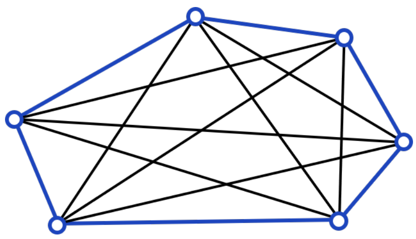
8. What is the coefficient of $x^3y^4z^5$ in the expansion of $(x+y+z)^{12}$? How many terms does this expansion have?

9. Find the number of triples (A, B, C) of subsets of $\{1, 2, 3, 4, 5\}$ such that $A \subseteq B \subseteq C$.

$*$ 10. (2022 AMC12) Suppose that n cards numbered $1, 2, 3, \ldots, n$ are arranged in a row. The task is to pick them up in numerically increasing order, working repeatedly from left to right. In the example below with $n = 13$, cards $1, 2, 3$ are picked up on the first pass, 4 and 5 on the second pass, 6 on the third pass, $7, 8, 9, 10$ on the fourth pass, and $11, 12, 13$ on the fifth pass. For how many of the $n!$ possible orderings of the cards will the n cards be picked up in exactly two passes?

7	11	8	6	4	5	9	12	1	13	10	2	3

$*$ 11. All the diagonals of a convex polygon of n-sides are drawn. Suppose that no three diagonals are concurrent and no two parallel. How many intersections do the diagonals make with each other?

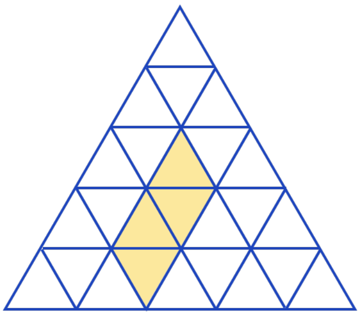


$*$ 12. (2023 AIME) Find the number of collections of 16 distinct subsets of $\{1, 2, 3, 4, 5\}$ with the property that for any two subsets X and Y in the collection, $X \cap Y \neq \emptyset$.

$*$ 13. (2023 AIME) Find the number of ways to place the integers 1 through 12 in the 12 cells of a 2×6 grid so that any two cells sharing a side, the difference between the numbers in those cells is NOT divisible by 3. The following table shows one way of doing this.

1	3	5	12	10	8
2	4	6	11	9	7

$**$ 14. We divide an equilateral triangle of side-length n into small equilateral triangles of side-length 1 by $n-1$ lines in each direction parallel to the sides. The resulting figure for $n = 4$ is shown in the figure. Count the number of paralleograms formed.



15. Use combinatorics to prove that
$$1+2+3+\cdots+n=\binom{n+1}{2}.$$

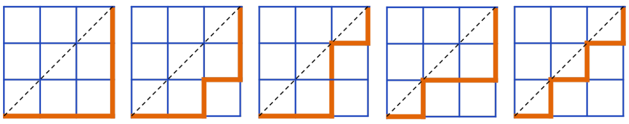
16. Use combinatorics to find a formula for $1^3+2^3+3^3+\cdots+n^3$.

$*$ 17. Prove that
$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2=\binom{2n}{n}.$$

$*$ 18. Compute the expression:
$$\binom{\binom{3}{2}}{2}+\binom{\binom{4}{2}}{2}+\binom{\binom{5}{2}}{2}+\cdots+\binom{\binom{n}{2}}{2}.$$

19. Listed below are six different types of objects. Show that all of them have the same count! This common count is known as the n-th Catalan number.

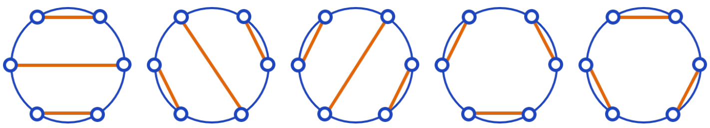
- Ways to go from $(0,0)$ to (n,n) by only increasing x coordinate by 1, or y coordinate by 1 (but not both) that goes above the line $y=x$.



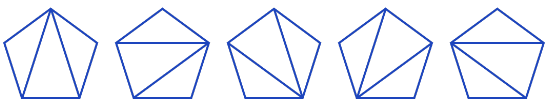
- Ways to place n pairs of brackets in a way that all the open brackets has a corresponding closed bracket.

((()))   ()(())   (())()   (()())   ()()()

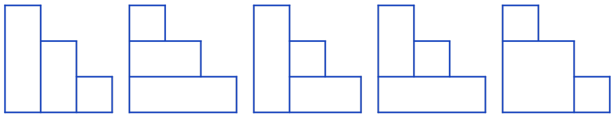
- Ways to pair up $2n$ equally spaced points on the circle with n chords so that no two chords intersect.



- $*$ Ways to draw diagonals inside a regular $(n+2)$-gon so that every region formed is a triangle.



- $*$ Ways to tile a stairstep shape of height n with n rectangles.



- $*$ Tree diagrams with $2n+1$ vertices where n of them has exactly 2 children each and the rest has no children.