Solutions to H_5

IMO Training for 2023

9 June 2023

Problem 1. Let there be five points of integer coordinates on the xy-plane. Show that at least one of the midpoints of the 10 possible segments determined by these points also have integer coordinates.

Solution. (Pigeonhole) Note that the midpoint of the lattice points (x_1, y_1) and (x_2, y_2) is a lattic point if and only if x_1, x_2 have same pairity, and y_1, y_2 also have the same pairity. Depending on the pairities, we can classify a lattice point (x, y) on the xy-plane into four possible types:

- Type 1: x is even and y is even,
- Type 2: x is even and y is odd,
- Type 3: x is odd and y is even,
- Type 4: x is odd and y is odd.

So, if there are five lattice points, by pigeonhole principle, two of these points have the same type and thus their midpoint will also be a lattice point. □

Problem 2. Let A be a subset of $\{1, 2, 3, \dots, 100\}$ such that $\gcd(x, y) > 1$ for any two distinct elements x and y of A. What is the maximum possible size of A?

Solution. (Pigeonhole) We will show that $\max(|A|) = 50$.

- We will show that $|A| = 50$ is possible. Simply take $A = \{2, 4, 6, \dots, 100\}$. Then, A satisfies the desired property and $|A| = 50$.
- We will now show $|A| \leq 50$ for all A satisfying the conditions in the problem. Partition $\{1, 2, 3, \dots, 100\}$ into 50 sets of the form $\{n, n+1\}$ $(n = 1, 3, 5, \dots, 99)$. If $|A| > 50$, by the Pigeonhole principle, there are two elements of A lying in $\{n, n+1\}$ for some $n = 1, 3, 5, \dots, 99$. But, $\gcd(n, n+1) = 1$, contradiction.

□

Problem 3. A polygon is called a *jigsaw-gon* if it satisfies the following properties:

- all the side-lengths are equal,
- every interior angle is either 90° or 270°.

Show that the number of vertices of a jigsaw-gon is always divisible by 4.

Solution. (Bijection) Think of the perimeter of the jigsaw-gon as a closed directed path. By rotating and scaling the jigsaw-gon if necessary, we can assume that we only travel in the directions *up*, *down*, *left*, or *right* and that the side length is 1. Let U, D, L and R be the number of times that we travel up, down, left and right respectively. We will show that

$$|U| = |D| = |L| = |R|. \tag{♠}$$

Proving this will immediately imply the conclusion.

- Note that total displacement in the vertical direction is equal to $|U| - |D|$ and the total displacement in the horizontal direction is equal to $|R| - |L|$. Since the path is closed, this implies that $|U| = |D|$ and $|R| = |L|$.
- Next, note that there is a bijection between the horizontal segments and the vertical segments. (We can describe this explicitly if we want: map each $e \in L \cup R$, to $e' \in U \cup D$ where e and e' are consecutive edges of the jigsaw-gon, and e' follows e in the path. This map can be easily verified to be a bijection.) Therefore,
$$|U| + |D| = |L| + |R|.$$

Combining the results of the two bullet points proves (♠). □

> **Problem 4.** Construct an injective function from $\mathbb{Q}$ into $\mathbb{N}$. Here, $\mathbb{Q}$ is the set of all rationals and $\mathbb{N}$ is the set of all positive integers.

Solution. (Bijection) In fact, we can construct a bijection between $\mathbb{Q}$ and $\mathbb{N}$. First, note that if A_0, A_1, A_2, ... are finite and pairwise disjoint sets, it is easy to see that there is a bijection between $\mathbb{N}$ and $A_0 \cup A_1 \cup A_2 \cup \cdots$. Now, let $A_0 = \{0\}$ and

$$A_k = \text{set of all reduced rationals with denominator } k.$$

for each $k = 1, 2, \ldots$. For example, $A_4 = \{\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{3}{4}\}$. Obviously, A_0, A_1, A_2, ... are finite and pairwise disjoint. Since $\mathbb{Q} = A_0 \cup A_1 \cup A_2 \cup \cdots$, we are done. □

Solution. (Injection) We shall construct $f : \mathbb{Q} \to \mathbb{N}$ as follows. Let $f(0) = 0$. For each non-zero reduced fraction $p/q \in \mathbb{Q}$, let $f(p/q)$ to be the $(p + q + 1)$-digit number

$$x\,\underbrace{333\ldots3}_{|p| \text{ threes}}\underbrace{444\ldots4}_{|q| \text{ fours}}$$

where $x = 1$ if $p/q > 0$ and $x = 2$ if $p/q < 0$. Then, f is injective: if $f(p_1/q_1) = f(p_2/q_2)$ for reduced fractions p_1/q_1 and p_2/q_2, we have

$$x_1\,\underbrace{333\ldots3}_{|p_1| \text{ threes}}\underbrace{444\ldots4}_{|q_1| \text{ fours}} = x_2\,\underbrace{333\ldots3}_{|p_2| \text{ threes}}\underbrace{444\ldots4}_{|q_2| \text{ fours}}$$

where x_1 and x_2 are determined by the sign of p_1/q_1 and p_2/q_2 respectively. Then, we have $x_1 = x_2$, $|p_1| = |p_2|$ and $|q_1| = |q_2|$. $x_1 = x_2$ tells us that p_1/q_1 and p_2/q_2 have the same sign, and hence the latter two equalities imply that $p_1/q_1 = p_2/q_2$. □

> **Problem 5.** A 6×6 grid is tiled with dominoes. Show that there exists a line cutting through a grid that does not go through any of the dominoes.

Solution. (Pigeonhole) The critical observation is that any horizontal or vertical line passes through an even number of dominoes. Indeed, any horizontal line l (resp. vertical) line cuts the grid into two parts one of which is of the form $n \times 6$ (resp. $6 \times n$) for some positive integer n. Note that this part contains even number of cells. Thus, if l goes through odd number of dominoes, the number of remaining cells in the $n \times 6$ (resp. $6 \times n$) part will be odd, which contradicts the fact that it is tiled with dominoes. Hence, any horizontal or vertical line passes through an even number of dominoes.

Now, there are 10 horizontal/vertical lines across the board and there are 18 dominoes covering the board. So, by pigeonhole principle, there exists a line passing through at most one domino. But, l cannot pass through exactly one domino. So, it must be the case that l does not pass through any dominoes. $\square$