Solutions to P_3 and H_3

IMO Training for 2023

3 June 2023

Problem 1. In a society of finite number of people, some people are friends with each other. Friendship is mutual, that is if A is friends with B, then B is also friends with A. Suppose that everyone has at most k friends. Prove that it is possible to separate these people into $k+1$ rooms so that no one knows each other in their own room.

Solution. (Greedy) The idea is to greedily assign the rooms. Prepare $k+1$ empty rooms and line up the people in a queue. Let P be the person at the front of the queue. Since P knows less than $k+1$ people, we can ensure that there is a room in which P knows no one. Send P into that room and repeat until there is no one left in the queue. $\square$

Problem 2. (Iran MO 2nd round/P2) There are n points in the plane such that no three of them are collinear. Prove that the number of triangles of area 1 formed by these points is at most $\frac{2}{3}(n^2-n)$.

Solution. (Double counting) Let $\mathcal{A}$ be the set of all area one triangles, present in the given configuration. Let $\mathcal{B}$ be the set of all $\binom{n}{2}$ unordered pairs of the given n points. Now, count the pairs $(T, \{p,q\})$ where the pair of points $\{p,q\} \in \mathcal{B}$ is two of the vertices of triangle $T \in \mathcal{A}$. Since every triangle has three edges,

$$\text{number of pairs} = 3|\mathcal{A}|.$$

Now, note that for any pair of points $\{p,q\} \in \mathcal{B}$, there are at most four points x in our configuration for which xpq is a triangle of area 1 because no three points are collinear. Therefore,

$$\text{number of pairs} \leq 4|\mathcal{B}|.$$

Therefore,

$$3|\mathcal{A}| \leq 4|\mathcal{B}| \qquad \Longrightarrow \qquad |\mathcal{A}| \leq \frac{2}{3}(n^2-n).$$

$\square$

Problem 3. (Canada MO 2012/P4) A number of robots are placed on the squares of a finite, rectangular grid of squares. A square can hold any number of robots. Every edge of each square of the grid is classified as either passable or impassable. All edges on the boundary of the grid are impassable. You can give any of the commands up, down, left, or right.
All of the robots then simultaneously try to move in the specified direction. If the edge adjacent to a robot in that direction is passable, the robot moves across the edge and into the next square. Otherwise, the robot remains on its current square. You can then give another command of up, down, left, or right, then another, for as long as you want. Suppose that for any individual robot, and any square on the grid, there is a finite sequence of commands that will move that robot to that square. Prove that you can also give a finite sequence of commands such that all of the robots end up on the same square at the same time.

Solution. (RUST) It suffices to show that any two robots R and S can meet together at a square. Draw the shortest path P between the squares that R and S are standing. Give the sequence of commands moving R along P. When R reaches the end of P, keep giving the commands as if R is at the beginning of P, and repeat. We will show that this process eventually stops, after which R and S shall coincide.

Note that S must stop at some point throughout this sequence of Indeed, if S stops at some point, the length of the shortest path between R and S will decrease. But, we claim that S must stop at some point eventually. Otherwise, S will move in a periodic pattern for infinitely many times which is not possible since S lies inside a bounded rectangular grid. So, the length of the shortest path between R and S is a monovariant that keeps decreasing, and hence the process must stop eventually. $\square$

Problem 4. In a certain committee, each member belongs to exactly three subcommittees, and each subcommittee has exactly three members. Prove that the number of members equals to the number of subcommittees.

Solution. (Double counting) Let $\mathcal{A}$ be the set of all subcommittees and $\mathcal{B}$ be the set of all members of the committee. Count all the pairs (S, M) of a subcommittee $S \in \mathcal{A}$ and a member $M \in \mathcal{B}$ such that M is a member of S. Since each member belongs to exactly three subcommittees,

$$\text{number of pairs} = 3|\mathcal{B}|.$$

Since each subcommittee contains exactly three members,

$$\text{number of pairs} = 3|\mathcal{A}|.$$

Hence, $|\mathcal{A}| = |\mathcal{B}|$. $\square$

Problem 5. Let G be a graph whose minimum degree is δ. Show that G contains a path with $\delta + 1$ vertices.

Solution. (RUST/Greedy) Let P be the longest path in G and let v be one of its ending vertices. Then, v is not joined with any vertex outside of P because otherwise, P cannot be the longest path. Thus, all neighbours of v lie on P. Since v has δ neighbors, P must contain at least $\delta + 1$ vertices (v and neighbors of v). $\square$

Problem 6. There are $n \geq 2$ students in a school some of whose are friends with each other. Friendship is mutual i.e. if A is friends with B, then B is also friends with A. The class teacher is going to separate them into two rooms. A student will be *sad* if at least half of his friends are in the room different from his. Show that the teacher can make every student sad.

Solution. (RUST) Start putting all the students in one room and leave the other room empty. Follow the following RUST algorithm:

"Whenever a student is not sad, move them to the opposite room."

We will show that this algorithm eventually terminates after which all students will be sad. Indeed, let C be the number of pairs of students (x, y) where x and y are in different rooms and x and y are friends. Whenever a student is moved, C increases. Since only finite amount of configurations is possible, C cannot increase forever, and thus our algorithm must terminate. □

Problem 7. Suppose 5000 distinct points in the plane are given such that no four points are collinear. Show that it is possible to select 100 of the points for which no three points are collinear.

Solution. (RUST/Greedy + double counting) Start with an empty set S of points. We follow the following greedy algorithm:

"Put a point in S if it is not collinear with two of the points that are already inside S."

Suppose that $|S| = k$ at the end of the algorithm. Let $\mathcal{A}$ be the set of points outside of S, and $\mathcal{B}$ be the set of unordered pairs of points inside S. We shall count the pairs $(X, \{P, Q\})$ of points $X \in \mathcal{A}$ and $\{P, Q\} \in \mathcal{B}$ so that X, P, Q are collinear. Since we cannot put more points into S at the end, for any point X outside of S, there exist a pair of points $\{P, Q\}$ insdie S so that P, Q, X is collinear. Therefore,

$$\text{number of pairs} \geq |\mathcal{B}|.$$

Since no four given points are collinear,

$$\text{number of pairs} \leq |\mathcal{A}|.$$

Therefore, $|\mathcal{A}| \leq |\mathcal{B}|$ which gives us

$$5000 - k \leq \binom{k}{2}.$$

Solving this (for example, using quadratic formula) gives us $k \geq 100$. □

Problem 8. (IMO 1989/P3) Let n and k be positive integers and let $\mathcal{S}$ be a set of n points in the plane such that

- no three points of $\mathcal{S}$ are collinear, and
- for every point P of $\mathcal{S}$, there are at least k points equidistant from P.

Prove that $k < \frac{1}{2} + \sqrt{2n}$.

Solution. (Double counting) Let $\mathcal{A}$ be the set of unordered pairs of points in $\mathcal{S}$. We shall count the number of pairs of the form $(P, \{Q, R\})$ where $P \in \mathcal{S}$, $\{Q, R\} \in \mathcal{A}$ and $PQ = PR$. Since every point of

$\mathcal{S}$ has at least k equidistant points form it, we have

$$\text{number of pairs} \geq \binom{k}{2}|\mathcal{S}|.$$

Since no three point of $\mathcal{S}$ are collinear, for any $\{Q, R\} \in \mathcal{A}$, there are at most 2 points $P \in \mathcal{S}$ such that $PQ = PR$ because such points P lie on the perpendicular bisector of QR. Hence,

$$\text{number of pairs} \leq 2|\mathcal{A}|.$$

So, $\binom{k}{2}|\mathcal{S}| \leq 2|\mathcal{A}|$. Substituting $|\mathcal{S}| = n$, $|\mathcal{A}| = n(n-1)/2$ and solving for k gives us

$$k < \frac{1}{2} + \sqrt{2n}.$$

$\square$