Solutions for P_2 and H_2

IMO Training for 2023

24 May 2023

Problem 1. Aliens abducted 100 mathematicians and put them into 100 separate rooms. Each room has a surveillance camera and each mathematician can see the 99 other mathematician's rooms except their own. Each room is painted in red or in blue, but the colour of the paint can only be seen in camera, not by naked eye. Then, the alien overlord makes an public announcement that at *least one of the rooms is painted blue, and that whoever that can figure out (with proof) the colour of their own room will be sent back to Earth!* Starting from that day, the alien overlord will privately talk to each mathematician once per day, asking for the proof (so they cannot just guess the colour). Suppose that every mathematician is perfectly smart i.e. they will know if such proof exists and will try to give it as soon as possible. Show that all mathematicians will eventually return to Earth.

Solution. For each positive integer $k \in \{1, 2, \dots, 100\}$, let $P(k)$ be the following statement:

If there are exactly k rooms painted in blue, then mathematicians in blue rooms will escape in the k-th day while mathematicians in red rooms will escape in $(k+1)$-th day.

Since the alien overlord announced that there is at least one blue room, $P(1)$ is true: the mathematician in blue room will immediately know the colour of his room, and following his escape, other mathematicians will also escape the next day.

Now, assume that $P(k)$ is true for some $1 \le k \le 99$ and suppose that exactly $k+1$ rooms are painted blue. Then, each mathematician M in a blue room can see exactly k blue rooms. So, they can deduce that the number of blue rooms is either k or $k+1$. But, since we assumed $P(k)$, mathematician M can simply wait for k days to see if the number of rooms is exactly k or not. Thus, on the $(k+1)$-th day, he can deduce that his room is blue and consequently, other mathematicians in red rooms will escape the next day. □

Problem 2. (14-15 puzzle) You are given a 4×4 grid in which 15 of them contain tiles with a number from $1, \dots, 15$ written in them and one empty tile. In a move you can slide a tile adjacent to the empty square. Is it possible, after some sequence of moves, to swap the positions of 14 and 15 while keeping other tiles fixed?

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Solution. We will show that the puzzle is impossible. Suppose to the contrary that it is solvable. The idea is to find two different alternating-variants. First variant is simple. If give the cells a blue-white checkerboard colouring, the colour of the empty cell will be alternating. Hence, we would have to make an even number of moves to finish the puzzle.

For the second variant, put the tile 16 in the empty square. Instead of sliding a square k into the empty cell, we will simply regard it as swapping the positions of k and 16. Now, in each position of the game, consider the permutation σ of $1, 2, \ldots, 16$ generated by reading the numbers in the cells row by row. Then, by what we have seen in lecture 3, the sign of σ (i.e. whether σ has even or odd number of inversions) will change with each move. But, sign of σ in the starting and ending positions are different because they differ by a single swap. Hence, we would have to make an odd number of moves to finish the puzzle, contradiction. □

Problem 3. (British MO 2003) Alice and Barbara play a game with a pack of $2n$ cards, on each of which is written a positive integer. The pack is shuffled and the cards laid out in a row, with the numbers facing upwards. Alice starts, and the girls take turns to remove one card from either end of the row, until Barbara picks up the final card. Each girl's score is the sum of the numbers on her chosen cards at the end of the game. Prove that Alice can always obtain a score at least as great as Barbara's.

Solution. Colour the cards alternatively in blue and white. Then, the colours of the cards at the ends of the row are different in Alice's turn while they have the same colour in Barbara's turn. Therefore,

- If Alice picks up a blue card, Barbara is forced to pick up a white card in her turn.
- If Alice picks up a white card, Barbara is forced to pick up a blue card in her turn.

Hence, if Alice keeps picking up blue cards, she will end up collecting all blue cards while Barbara gets all the white cards and vice versa. Thus, Alice can simply calculate the sum of the numbers on blue cards and compare it with that of the white cards. □

Problem 4. (USAMO 2002/P1) Let S be a set with 2002 elements, and let N be an integer with $0 \leq N \leq 2^{2002}$. prove that it is possible to color every subset of S either black or white so that the following conditions hold:

- the union of any two white subsets is white,
- the union of any two black subsets is black,
- there are exactly N white subsets.

Solution. We will replace 2002 with m and show the conclusion for all positive integers m. This is obviously true if $m = 1$. Now, suppose we have already proven the conclusion for $m = 1, \ldots, k-1$ for some positive integer $k \geq 2$. Let $S = \{a_1, a_2, \ldots, a_{k+1}\}$ and $0 \leq N \leq 2^{k+1}$.

- We will first consider the case $0 \leq N \leq 2^k$. In this case, exactly N subsets of $S \setminus \{a_1\} = \{a_2, a_3, \ldots, a_{k-1}\}$ can be coloured in white so that union of two white subsets (of $S \setminus \{a_1\}$) is white and union of two black subsets is black. We may extend this colouring to subsets of S simply by colouring every subset containing a_1 to be black. Then, union of two white subsets is still white. Let B_1 and B_2 be two black subsets. If one of B_1, B_2 contains a_1, then $B_1 \cup B_2$ also contains a_1 and hence $B_1 \cup B_2$ is black. Otherwise, both B_1 and B_2 are black subsets of $S \setminus \{a_1\}$ and hence $B_1 \cup B_2$ is black.

- If $2^k < N \leq 2^{k+1}$, then by the above case, we can instead colour $2^{k+1} - N$ subsets white and N subsets black. Then, we can just switch the colour of each subsets which obviously does not change the fact that taking union does not change the colour.

□

Problem 5. There are three piles with n tokens each. In every step we are allowed to choose two piles, take one token from each of those two piles and add a token to the third pile. Using these moves, is it possible to end up having only one token?

Solution. Note that the number of tokens in each pile changes by either $+1$ or -1 in each move. Hence, the parity of the number of tokens in each pile changes with each move. Since all tokens have same parity (i.e. all even or all odd) in the beginning, they will still have the same parity after each move. Therefore, it is impossible to end up with 1 token in one of the piles and 0 tokens in the others.
Note: Other approaches of the similar spirit is also possible. For example, we can note that for any two piles, the difference of the number of tokens between them either increases by 2, decreases by 2 or stays the same. Hence, the parity of the difference doesn't change. □

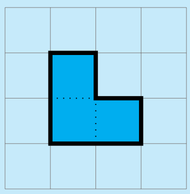
Problem 6. Cockroach Joey is sitting at the lower left corner of an $n \times n$ grid. Each second, Joey crawls to the adjacent square (two squares are adjacent if they share a side). Its goal is to to crawl through every square of the grid exactly once and end up at the upper right corner. Find all possible values of n for which Joey can achieve its goal.

Solution. We claim that the answer is all odd n.

- Suppose n is odd. Then, Joey can traverse the first row from left to right, then the second row from right to left, then the third row from left to right, $\ldots$, then the n-th row from left to right.
- Suppose n is even. Colour the board in white and blue alternatively as in a checkerboard. Then, Joey changes his colour with each move. Thus, if Joey visits all the squares exactly once, then he would have to make $n^2 - 1$ moves and thus the starting cell and ending cell have different colours. But, the lower left corner and the upper right corner have the same colour because they lie on the diagonal. So, it is impossible to finish at the upper-right corner.

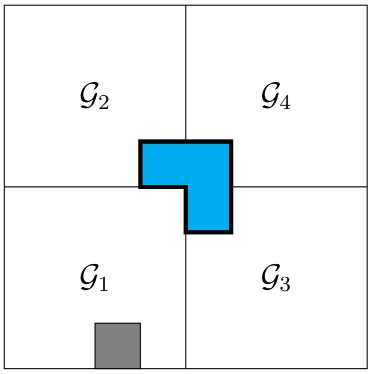
□

Problem 7. One of the 1×1 cells of a given $2^n \times 2^n$ grid is removed. Show that it is possible to tile the resulting grid using L-shaped tiles shown below.



You may rotate or reflect the tiles, the tiles shall not exceed the boundaries of the grid and may not overlap.

Solution. If $n = 1$, we can trivially do this. Suppose we know how to tile a $2^{n-1} \times 2^{n-1}$ board with one cell removed. We will show that we can tile a $2^n \times 2^n$ board $\mathcal{G}$ with one cell removed. Divide $\mathcal{G}$ into four $2^{n-1} \times 2^{n-1}$ grids $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$ where $\mathcal{G}_1$ has a missing cell. By inductive assumption, we can tile $\mathcal{G}_1$. Now, place an L-shaped tile at the center of $\mathcal{G}$ so that it intersects $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$ at one cell each. Then, again by inductive assumption, we can tile the remaining parts of $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$.



Note: If you missed the idea of placing one L-shaped tile at the center, we can still use another inductive argument to show that the $2^n \times 2^n$ grid with the $2^{n-1} \times 2^{n-1}$ subgrid at the top-right corner removed can be tiled by L-shaped tiles. It is also interesting to see that this problem gives a cool combinatorial way to prove $3 \mid 4^n - 1$ which is classically proven by noting that

$$4^n - 1 = (4-1)(4^{n-1} + 4^{n-2} + \cdots + 4^1 + 1).$$

□

Problem 8. Prove that we can rearrange the terms of the sequence $1, 2, 3, \ldots, 2023$ so that the average of any two distinct terms does not appear between them in the rearranged sequence.

Solution. Replace 2023 with n and we will prove by strong induction that such an arrangement is possible for the sequence $1, 2, 3, \ldots, n$. The cases $n = 1$ and $n = 2$ are trivial. Let $A = \{2, 4, 6, \ldots, 2\lfloor n/2 \rfloor\}$ be the set of even numbers in $1, 2, \ldots, n$ and $B = \{1, 3, 5, \ldots, 2\lfloor (n+1)/2 \rfloor - 1\}$ be the set of odd numbers in $1, 2, \ldots, n$. For convenience, if a sequence satisfies the property that the average of any two terms do not lie between them, then we shall call the sequence *AP-free*.

By inductive assumption, we know how to rearrange the terms $1, 2, \ldots, \lfloor n/2 \rfloor$ in an AP-free sequence. By doubling each term, we can arrange the terms of A in an AP-free sequence. Similarly, by inductive assumption, we can rearrange the terms $1, 2, \ldots, \lfloor (n+1)/2 \rfloor$ in an AP-free sequence. By doubling each term and subtracting one from each term, we obtain an AP-free arrangement of the terms of B. Now, the desired AP-free arrangement of $1, 2, 3, \ldots, n$ follows by concatinating (i.e. writing one after another) the AP-free arrangement of A and the AP-free arrangement of B because the average between a term in A and a term in B is not an integer.

Note: It is tempting to approach this problem by ordinary induction. That is, by trying to show that we can insert the term n in the AP-free arrangement of the sequence $1, 2, 3, \ldots, n-1$. But, this approach does not work without further modifications because we can artificially construct an AP-free arrangement of $1, 2, 3, \ldots, n-1$ for which we cannot insert n. For example, if the AP-free arrangement generated by inductive assumption is $1, 4, 6, 5, 3, 2$, then there is no way to insert 7 into this sequence. □