Solutions for P_1 and H_1

IMO 2023

May 12, 2023

Problem 1. Let n be a positive integer. Positive real numbers $a_1, \ldots, a_n$ are written on a blackboard. In each move, Cory erases two numbers a and b from the board and writes $ab/(a+b)$. Show that after $n-1$ moves, the number left on the board is independent of Cory's choices.

Solution. [using invariant] Let S be the sum of reciprocals of the numbers written on the board. We will show that S does not change under Cory's operations. Indeed, suppose at some stage, the numbers written on the board are $b_1, b_2, \ldots, b_m$ where $m \le n$. If Cory picked b_i and b_j, then the change in the value of S is

$$-\frac{1}{b_i} - \frac{1}{b_j} + \frac{1}{b_i b_j/(b_i+b_j)} = -\frac{1}{b_i} - \frac{1}{b_j} + \frac{b_i+b_j}{b_i b_j} = 0$$

Hence, indeed S does not change after each move. Therefore, since $S = 1/a_1 + \cdots + 1/a_n$ at the start, the value of S at the end (when only 1 number is left) is also $1/a_1 + \cdots + 1/a_n$ as well. Thus, his last number must be equal to $1/(1/a_1 + \cdots + 1/a_n)$ which only depends on the starting numbers. □

Problem 2. (Canada MO 1994/P3) There are $2n+1$ lamps placed in a circle. Each day, some of the lamps change state (from on to off or off to on), according to the following rules. On the k-th day, if a lamp is in the same state as at least one of its neighbors, then it will not change state the next day. If a lamp is in a different state from both of its neighbors on the k-th day, then it will change its state the next day. Show that regardless of the initial states of each lamp, after some point none of the lamps will change state.

Solution. [using monovariant] Call a lamp L *good* if at least one neighboring lamp of L have the same state as L and call it *bad* otherwise. Note that once a lamp is good, it will always be good, and so the number of good lamps cannot decrease. Also note that if a lamp is bad, its state will change in the next round.

Since the number of lamps in the beginning is odd, there must be two consecutive lamps of the same state. Hence, the number of good lamps is non-zero at the start. We will show that whenever we have a good lamp and a bad lamp in our configuration, the number of good lamps strictly increases. Indeed, if we have both good and bad lamps, there exist two adjacent lamps L_1 and L_2 such that L_1 is good and L_2 is bad. Then, states of L_1 and L_2 are different (because L_2 is bad). After the next round, since L_2 changes its state while L_1 doesn't, L_1 and L_2 will then be in same state. Thus, the number of good lamps increases.

Hence, the number of good lamps will eventually become $2n+1$ and at this point, no lamp is going to change the state. □

Solution. (*due to Soe Lin Htet*) Suppose to the contrary that the lamps change their states forever. Then, at least one of the lamps (call it L) must have changed its state for infinitely many times. Let M be a neighbor of L. Then, we will show that M also changes its state infinitely many times. Indeed, if

M only changes its state finitely many times, then after some point, M will always be on or always be off. But, since L changes states infinitely many time, state of M and L will agree after some point after which L will not change anymore. Therefore, it must be the case that M changes its state infinitely many times.

This implies that every lamp changes their state infinitely many times. But, since $2n + 1$ is odd, there exist two consecutive lamps of the same state at the start. These lamps will never change their state, contradiction. □

Problem 3. (Hexachord Theorem) Consider $2n$ points equally spaced around a circle. Suppose that n of the points are coloured blue and the remaining n points red. We write down the distance between each pair of blue points in a list, from shortest to longest. We similarly write down the distance between each pair of red points in another list, from shortest to longest. Prove that the two lists of distances are identical (note that the same distance may occur more than once in a list).

Solution. [using invariant] Note that if n consecutive points are coloured red and the other n consecutive points blue, then the lists of distances are identical by symmetry. Now, suppose we have a configuration with identical blue distance list and red distance list. We will show that the distances will still be identical after two adjacent red point R and blue point B are switched.

The only change on the distances are those distances between pairs involving R or B. Pick any other point P and let Q be its reflection at the perpendicular bisectior of RB.

- If P and Q have same colour, then swapping R and B permutes the red distances, and blue distances separately.
- If P and Q have different colours, then swapping R and B changes two identical distances in red and blue lists to another two identical distances.

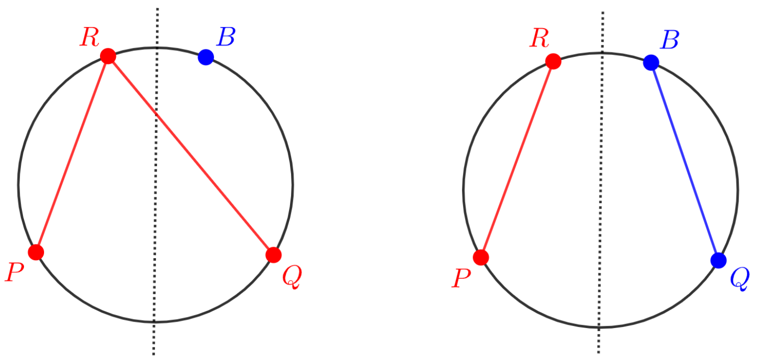

Figure 1: Tracking how swapping affects each distance (for red P)

In any case, switching R and B keeps the distance lists identical. Now, since any configuration can be reached from a symmetric one by swapping adjacent red and blue point, the distance lists will be identical for any configuration. □

Problem 4. Initially, m cells of an $n \times n$ grid, contains a *water source* and the rest are *empty*. In each second, every empty cell adjacent to at least two cells filled with water source also becomes a water source (two cells are adjacent if they share an edge). What is the smallest possible value of m so that the entire grid will eventually be filled with water source?
(Short version: How many water buckets do you need to fill a $n \times n$ minecraft pool?)

Solution. [using monovariant] Placing water sources along the diagonal fills up the entire grid eventually. Therefore, having $m = n$ water sources initially can fill up the grid. Hence, $\min(m) \leq n$.

Now, suppose m water sources can eventua lly fill the grid. We will show that $m \geq n$. In each second, place the newly created water sources one by one on the board. Note by checking cases that each time a new water source is placed, the perimeter of the union of cells containing water source never increases.

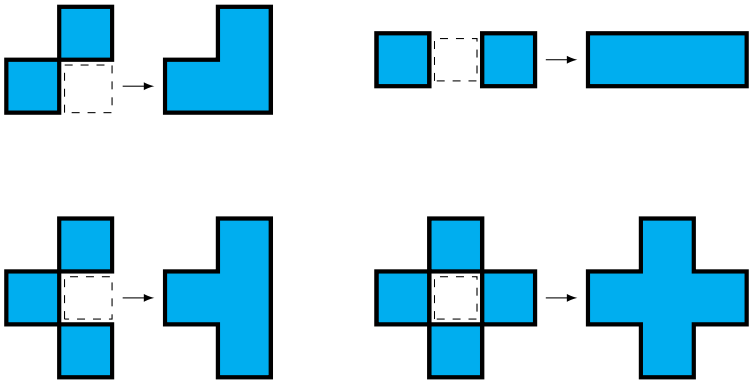

Figure 2: All possible ways for an empty square to become a water source

Therefore, the starting perimeter must be at least $4n$. Note that the actual starting perimeter is at most $4m$. Therefore, $4m \geq 4n$ and hence $m \geq n$. □

Problem 5. Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Three frogs Alice, Bob and Chris are sitting together at a point on the Euclidean plane. For each $1 \leq k \leq n$, at the k-th round, each of them jump a_k units in the direction of their own choice: north or east. Show that after all the rounds, the positions of Alice, Bob and Chris are collinear.

Solution. [using invariant] Construct the coordinate system so that the frogs initially sit on the origin and that positive x-axis points to the east while positive y-axis points to the north.

Then, whenever Alice makes a move in the k-th round, sum of her x and y coordinates increases by a_k. Therefore, after all the rounds, Alice lies on the straight line whose equation is

$$x + y = a_1 + a_2 + \cdots + a_n.$$

Similarly, Bob and Chris also lie on that line and hence their positions are collinear. □

Problem 6. In an online shooter game, 100 players participate in a match. Once the game starts, players can start to shoot and kill each other. Players come back to life after death (that is, a player can be killed more than once) and they cannot kill themselves. The *score* of each player starts with 1, and gets $+1$ for each kill and -2 for each death (negative scores are possible). At the end of the match, it was found that the sum of the scores of all the players is 0. Prove that no player got more than 100 kills.

Solution. [using monovariant] Note that whenever a kill happens, sum of all the scores of the players decreases by 1. Since the sum of all scores is 100 initially and sum of all scores is 0 at the end, total number of kills is exactly 100. Hence, no player got more than 100 kills. □

Problem 7. 2023 cups are placed upside down on a table. You can take two cups at a time and flip them. Is it possible to make every cup right-side up?

Solution. [using invariant] The answer is no. Note that whatever move we do, the number of upside down cups either increases by 2, decreases by 2 or stays the same. Therefore, the parity of the number of upside down cups does not change. Since we started with odd number of upside down cups, it is impossible to reach a situation with zero upside down cups. ☐

Problem 8. A non-zero real number is written in each cell of an $m \times n$ table. You are allowed to pick any row or column, and change the signs of every number lying in it. Show that it is possible make a sequence of moves so that sum of the entries in each row or column is non-negative.

Solution. [using monovariant] Consider the following algorithm:

Whenever you see a row/column whose sum is negative, flip all the signs in it.

Note that after each step of the algorithm, the sum of all the numbers on the board will increase. Since there are only finitely many possible configurations, this means that the algorithm must stop eventually. At the point where we cannot run our algorithm anymore, all of the rows and columns must have non-negative sum. ☐