# Problem Set 7

IMO Training for 2023

13 June, 2023

**Problems for this week**

Please try to collect at least [5♦]'s before the next problem solving session.
**Note:** the above [♦]-quota is mandatory for IMO team members.

[2♦] 1. Amy and Ben are going to play a game by turns. Before they start, they form a circle with 2001 other people. At every turn, they can remove one of their neighbors from the circle. The winner is the one who gets the other person out of the circle. If Amy starts, who has a winning strategy? *Note: The other* 2001 *people do not have their own turns.*

[3♦] 2. Problem 2

[3♦] 3. Problem 3

[4♦] 4. (USAMO 1999/P5) The Y2K-Game is played on a $1 \times 2000$ grid as follows. Two players Andrew and Bard take turns to write either a $\mathcal{S}$ or an $\mathcal{O}$ starting with Andrew. The first player who produces three consecutive boxes that spell $\mathcal{SOS}$ wins. If all boxes are filled without producing $\mathcal{SOS}$, then the game is a draw. Prove that Bard has a winning strategy.

**For Further Practice**

2♥ 5. $A$ and $B$ each get an unlimited supply of identical circular coins. $A$ and $B$ take turns placing the coins on a finite square table, in such a way that no two coins overlap and each coin is completely on the table (that is, it doesn't stick out). The person who cannot legally place a coin loses. Assuming at least one coin can fit on the table, prove that $A$ has a winning strategy.

2♥ 6. There is a pile of $m$ blue cards and another pile of $n$ red cards. Players $A$ and $B$ take turns to remove the cards start with $A$. In their turn, the player chooses a pile, and remove the cards from that pile as many as they want. Find all possible values of $m$ and $n$ for which $A$ has winning strategy.

3♥ 7. Let $N$ be a positive integer with at least 3 distinct prime factors. Amy and Ben take turns writing composite divisors of $N$ (including $N$) on a board, starting with Amy. The rule is that there may never appear two coprime numbers on the board, and there may never appear two numbers one of which divides the other. The first player who cannot make a move loses. Who has a winning strategy?

3♥ 8. A chess *king* is placed at the centre of a $5 \times 5$ board. Two players Alice and Bob take turns to move the king, starting with Alice. The king may be moved to any cell that shares a vertex with the current vertex, but it may not move to a cell that it has visited before (the centre of the board is regarded as visited). Who has the winning strategy?

3♥ 9. (SMMC 2018/A2) Ada and Byron play a game. First, Ada chooses a non-zero real number $a$ and announces it. Then Byron chooses a non-zero real number $b$ and announces it. Then, Ada chooses a non-zero real number $c$ and announces it. Finally, Byron chooses a quadratic polynomial whose three coefficients are $a$, $b$, $c$ in some order.

(a) Suppose that Byron wins if the quadratic polynomial has a real root and Ada wins otherwise. Determine which player has a winning strategy.

(b) Suppose that Ada wins if the quadratic polynomial has a real root and Byron wins otherwise. Determine which player has a winning strategy.

4♥ 10. (GQMO 2020/P5) Let $n$ and $k$ be positive integers such that $k \leq 2^n$. Banana and Corona are playing the following variant of the guessing game. First, Banana secretely picks an integer $x$ such that $1 \leq x \leq n$. Corona will attempt to determine $x$ by asking some questions, which are described as follows: in each turn, Corona chooses $k$ distinct subsets of $\{1, 2, \ldots, n\}$ and, for each chosen set $S$, asks the question "Is $x$ in the set $S$?". Banana then picks one of these $k$ questions and tells both the question and its answer to Corona, who can then start another turn.

Find all pairs $(n, k)$ such that, regardless of Banana's actions, Corona could determine $x$ in finitely many turns with absolute certainty.

4♥ 11. (IMOSL 2017/N2) Let $p \geq 2$ be a prime number. Eduardo and Fernando play the following game making moves alternately: in each move, the current player chooses an index $i$ in the set $\{0, 1, 2, \ldots, p-1\}$ that was not chosen before by either of the two players and then chooses an element $a_i$ from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Eduardo has the first move. The game ends after all the indices have been chosen .Then the following number is computed:

$$M = a_0 + a_1 10 + a_2 10^2 + \cdots + a_{p-1} 10^{p-1} = \sum_{i=0}^{p-1} a_i.10^i.$$

The goal of Eduardo is to make $M$ divisible by $p$, and the goal of Fernando is to prevent this.

Prove that Eduardo has a winning strategy.

4♥ 12. (IMO 2016/P6) There are $n \geq 2$ line segments in the plane such that every two segments cross and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it facing the other endpoint. Then he will clap his hands $n-1$ times. Every time he claps,each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

(a) Prove that Geoff can always fulfill his wish if $n$ is odd.

(b) Prove that Geoff can never fulfill his wish if $n$ is even.

4♥ 13. (IMO 2018/P4) Queenie and Horst play a game on a $20 \times 20$ chessboard. In the beginning the board is empty. In ever turn, Horst places a black knight on an empty square in such a way that his new knight does not attack any previous knights. Then, Queenie places a white queen on an empty square. The game gets finished when somebody cannot move.

Find the maximal positive $K$ such that, regardless of the strategy of Queenie, Horst can put at least $K$ knights on the board.

5♥ 14. (IMO 2006/P2) Let $P$ be a regular 2006-gon. A diagonal is called good if its endpoints divide the boundary of $P$ into two parts, each composed of an odd number of sides of $P$. The sides of $P$ are also called good.

Suppose $P$ has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of $P$. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

5♥ 15. (RMM 2008/P4) Consider a square of sidelength $n$ and $(n+1)^2$ interior points. Prove that we can choose 3 of these points so that they determine a triangle (possibly degenerate) of area at most $1/2$.

5♥ 16. (USAMO 2005/P5) Let $n$ be an integer greater than 1. Suppose $2n$ points are given in the plane, no three of which are collinear. Suppose $n$ of the given $2n$ points are colored blue and the other $n$ colored red. A line in the plane is called a balancing line if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side.

Prove that there exist at least two balancing lines.

5♥ 17. (IMOSL 2006/C3) Let $S$ be a finite set of points in the plane such that no three of them are on a line. For each convex polygon $P$ whose vertices are in $S$, let $a(P)$ be the number of vertices of $P$, and let $b(P)$ be the number of points of $S$ which are outside $P$. A line segment, a point, and the empty set are considered as convex polygons of 2, 1, and 0 vertices respectively. Prove that for every real number $x$

$$\sum_P x^{a(P)}(1-x)^{b(P)} = 1,$$

where the sum is taken over all convex polygons with vertices in $S$.

5♥ 18. (Canada MO 2019/P5) A 2-player game is played on $n \geq 3$ points, where no 3 points are collinear. Each move consists of selecting 2 of the points and drawing a new line segment connecting them. The first player to draw a line segment that creates an odd cycle loses. (An odd cycle must have all its vertices among the $n$ points from the start, so the vertices of the cycle cannot be the intersections of the lines drawn.) Find all $n$ such that the player to move first wins.

5♥ 19. (Canada MO 2017/P5) There are 100 circles of radius one in the plane. A triangle formed by the centres of any three given circles has area at most 2017. Prove that there is a line intersecting at least three of the circles.

6♥ 20. (IMOSL 2021/C6) A hunter and an invisible rabbit play a game on an infinite square grid. First the hunter fixes a colouring of the cells with finitely many colours. The rabbit then secretly chooses a cell to start in. Every minute, the rabbit reports the colour of its current cell to the hunter, and then secretly moves to an adjacent cell that it has not visited before (two cells are adjacent if they share an edge). The hunter wins if after some finite time either:

- the rabbit cannot move; or
- the hunter can determine the cell in which the rabbit started.

Decide whether there exists a winning strategy for the hunter.

6♥ 21. (IMOSL 2014/C7) Let $M$ be a set of $n \geq 4$ poitns in the plane, no three of which are collinear. Initially, these points are connected with $n$ segments so that each point in $M$ is the endpoint of exactly two segments. Then, at each step, one may choose two segments $AB$ and $CD$ sharing a common interior point and replace them by the segments $AC$ and $BD$ if none of them is present at this moment. Prove that it is impossible to perform $n^3/4$ or more such moves.

7♥ 22. (IMO 2017/P3) A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, $A_0$, and the hunter's starting point, $B_0$ are the same. After $n-1$ rounds of the game, the rabbit is at point $A_{n-1}$ and the hunter is at point $B_{n-1}$. In the $n^{\text{th}}$ round of the game, three things occur in order:

i. The rabbit moves invisibly to a point $A_n$ such that the distance between $A_{n-1}$ and $A_n$ is exactly 1.

ii. A tracking device reports a point $P_n$ to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between $P_n$ and $A_n$ is at most 1.

iii. The hunter moves visibly to a point $B_n$ such that the distance between $B_{n-1}$ and $B_n$ is exactly 1.

Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after $10^9$ rounds, she can ensure that the distance between her and the rabbit is at most 100?

7♥ 23. (IMO 2011/P2) Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A windmill is a process that starts with a line $\ell$ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the pivot $P$ until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, $Q$, takes over as the new pivot, and the line now rotates clockwise about $Q$, until it next meets a point of $\mathcal{S}$. This process continues indefinitely.

Show that we can choose a point $P$ in $\mathcal{S}$ and a line $\ell$ going through $P$ such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

8♥ 24. (IMO 2020/P6) Prove that there exists a positive constant $c$ such that the following statement is true: Consider an integer $n > 1$, and a set $\mathcal{S}$ of $n$ points in the plane such that the distance between any two different points in $\mathcal{S}$ is at least 1. It follows that there is a line $\ell$ separating $\mathcal{S}$ such that the distance from any point of $\mathcal{S}$ to $\ell$ is at least $cn^{-1/3}$.

(A line $\ell$ separates a set of points S if some segment joining two points in $\mathcal{S}$ crosses $\ell$.)