Problem Set 6

IMO Training for 2023

7 June, 2023

Problems for this week

Please try to collect at least $\boxed{5\blacklozenge}$'s before the next problem solving session.
Note: the above $\boxed{\blacklozenge}$-quota is mandatory for IMO team members.

$\boxed{2\blacklozenge}$ 1. Let k be a positive integer. Find all positive integers n such that the $n \times n$ grid can be tiled with $1 \times k$ tiles. As usual, tiles shall not exceed the boundary of the grid or overlap, and You can rotate the tiles.
[Hint:]

$\boxed{3\blacklozenge}$ 2. Each edge of the complete graph K_n with $n \geq 3$ is given a direction to construct a directed graph. Suppose that there is a directed cycle passing through every vertex exactly once. Show that there are three vertices, also forming a directed cycle.
[Hint:]

$\boxed{3\blacklozenge}$ 3. Prove that a cube cannot be cut into a finite number of smaller cubes no two of which are congruent to each other.
[Hint:]

$\boxed{4\blacklozenge}$ 4. Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of these stones and moves it to the right by at most k squares (the stone should say within the board). Sisyphus' aim is to move all n stones to square n.

Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

turns. (As usual, $\lceil x \rceil$ stands for the least integer not smaller than x.)
[Hint:]

For Further Practice

$\boxed{2♥}$ 5. Let G be a connected graph with m edges. Initially, there are $m-1$ frogs sitting at a vertex. Every second, whenever a vertex v has at least $\deg(v)$ frogs, $\deg(v)$ frogs jump from v to its neighboring vertices, one for each neighbor. Show that this process terminates eventually.
[Hint:]

$\boxed{2♥}$ 6. Real numbers are written inside the cells of a finite grid so that there is one number in each cell and the number written in each cell that is not on the boundary is average of the numbers written in the neighboring cells (two cells are neighboring if they share a veretx). If all the numbers on the boundary are equal, show that every number written in the cell must be equal.
[Hint:]

$\boxed{2♥}$ 7. Let Ω be a set of points in the plane. Each point in Ω is a midpoint of two other points in Ω. Show that Ω is infinite.

$\boxed{3♥}$ 8. (IMOSL 2011/C2) Suppose that 1000 students are standing in a circle. Prove that there exists an integer k with $100 \leq k \leq 300$ such that in this circle, there exists a contiguous group of $2k$ students, for which the first half contains the same number of girls as the second half.

$\boxed{3♥}$ 9. (EGMO 2012/P2) Let n be a positive integer. Find the greatest possible integer m in terms of n with the following property: a table with m rows and n columns can be filled with real numbers in such a manner that for any two different rows $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$, we have

$$\max\{|a_1 - b_1|, |a_2 - b_2|, \ldots, |a_n - b_n|\} = 1.$$

$\boxed{4♥}$ 10. (Estonia TST 2021/P4) There are $2n$ rays marked in a plane, with n being a natural number. Given that no two marked rays have the same direction and no two marked rays have a common initial point, prove that there exists a line that passes through none of the initial points of the marked rays and intersects with exactly n marked rays.

$\boxed{4♥}$ 11. (ELMO SL 2012/C3) Find all ordered pairs of positive integers (m, n) for which there exists a set $\mathcal{C}$ of colours and an assignment of colours to each of the mn cells of an $m \times n$ grid in such a way that for every colour $c \in \mathcal{C}$, and a cell S of colour c, exactly two adjacent cells of S also have colour c. (Here, two cells are adjacent if they share a common edge).

$\boxed{4♥}$ 12. (APMO 2007/P5) A regular (5×5)-array of lights is defective, so that toggling the switch for one light causes each adjacent light in the same row and in the same column as well as the light itself to change state, from on to off, or from off to on. Initially all the lights are switched off. After a certain number of toggles, exactly one light is switched on. Find all the possible positions of this light.

$\boxed{5♥}$ 13. Twenty three friends want to play football. For this, they choose one of them to be a referee and the others split into two teams of 11 persons each. They want to do this so that the total weight of each team is the same. We know that they all have integer weights and that, regardless of who is the referee, it is possible to make the two teams. Prove that they all have the same weight.

$\boxed{5♥}$ 14. (USAMO 2005/P5) Let $n > 1$ be an integer. Suppose $2n$ points are given in the plane, no three of which are collinear. Suppose n of the given $2n$ poitns are coloured blue and the other n are coloured red. A line in the plane is called a *balancing line* if it passes through one blue and one red point, and for each side of the line, the number (possibly zero) of blue points on that side is equal to the number of red points on the same side.

Prove that there exist at least two balancing lines.

7♥ 15. (IMO 2007/P3) In a mathematical competition, some competitors are friends; friendship is always mutual. Call a group of competitors a *clique* if each two of them are firends. The number of members in a clique is called its size. It is known that the size of the largest clique(s) is(are) even. Prove that the competitors can be arranged in two rooms such that the size of the largest cliques in one room is the same as the size of the largest cliques in the other room.

8♥ 16. (China TST3 2017/P6) Every cell of a 2017×2017 grid is colored either black or white, such that every cell has at least one side in common with another cell of the same color. Let V_1 be the set of all black cells, V_2 be the set of all white cells. For set $V_i (i = 1, 2)$, if two cells share a common side, draw an edge with the centers of the two cells as endpoints, obtaining graphs G_i. If both G_1 and G_2 are connected paths (no cycles, no splits), prove that the center of the grid is one of the endpoints of G_1 or G_2.