Problem Set 5

IMO Training for 2023

31 May, 2023

Problems for this week

Please try to collect at least $\boxed{5\blacklozenge}$'s before the next problem solving session.
Note: the above $\boxed{\blacklozenge}$-quota is mandatory for IMO team members.

$\boxed{2\blacklozenge}$ 1. Let there be five points of integer coordinates on the xy-plane. Show that at least one of the midpoints of the 10 possible segments determined by these points also have integer coordinates.
[Hint:]

$\boxed{2\blacklozenge}$ 2. Let A be a subset of $\{1, 2, 3, \ldots, 100\}$ such that $\gcd(x, y) > 1$ for any two distinct elements x and y of A. What is the maximum possible size of A?
[Hint:]

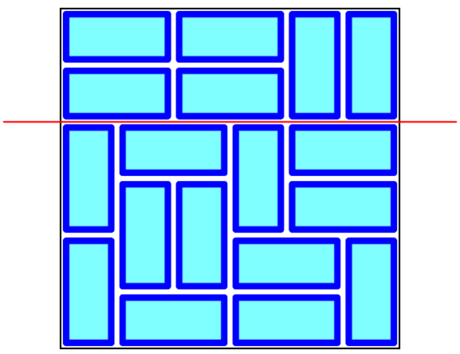
$\boxed{3\blacklozenge}$ 3. A polygon is called a *jigsaw-gon* if it satisfies the following properties:

- all the side-lengths are equal,
- every interior angle is either $90°$ or $270°$.

Show that number of vertices of a jigsaw-gon is always divisible by 4.
[Hint:]

$\boxed{3\blacklozenge}$ 4. (Countability of Integers) Construct an injective function from $\mathbb{Q}$ into $\mathbb{N}$. Here, $\mathbb{Q}$ is the set of all rationals and $\mathbb{N}$ is the set of all positive integers.
[Hint:]

$\boxed{4\blacklozenge}$ 5. A 6×6 grid is tiled with dominoes. Show that there exists a line cutting through the grid that does not go through any of the dominoes. (Just like in the figure below.)



[Hint:]

For Further Practice

2♥ 6. Without using the binomial theorem, show that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0$$

for all positive integers n.

[Hint:]

2♥ 7. Prove that the decimal representations of rational numbers are either finite or eventually periodic.

[Hint:]

2♥ 8. Let $x_1, x_2, \ldots, x_{2023}$ be positive integers and $y_1, y_2, \ldots, y_{2023}$ be their permutation. Prove that the product

$$(x_1 - y_1)(x_2 - y_2)(x_3 - y_3) \cdots (x_{2023} - y_{2023})$$

is always even.

2♥ 9. Let $\alpha > 0$ be an irrational number. Prove that for any two real numbers a and b with $a < b$, there exists a positive integer n such that

$$a < \{n\alpha\} < b$$

where $\{n\alpha\}$ denotes the fractional part of $n\alpha$.

3♥ 10. Let there be 7 lines on the plane, no two of which are parallel. Show that we can select two of these lines so that the (acute) angle between them is less than $26°$.

[Hint:]

3♥ 11. Let f_n be the number of ways to tile a $1 \times n$ rectangle with 1×2 dominoes and 1×1 square tiles.

$$f_1 - f_2 + f_3 - f_4 + \cdots + f_{2023} = f_{2022} + 1.$$

Bonus: 2♥ Show that f_n is the $(n+1)$-th Fibonacci number i.e. the $(n+1)$-th term of the sequence $1, 1, 2, 3, 5, 8, 13, \ldots$.

3♥ 12. (IMO 1964/P4) Seventeen people correspond by mail with one another – each one with all the rest. In their letters, only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to one another about the same topic.

[Hint:]

3♥ 13. (From lecture 10) Show that whenever the edges of the complete graph K_6 is coloured in either red or blue, there always exist at least 2 monochromatic triangles.

[Hint:]

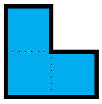
4♥ 14. Let A be a subset of $\{1, 2, 3, \ldots, 100\}$. What is the maximum possible size of $|A|$ if no element of A divides another.

[Hint:]

4♥ 15. (EGMO 2020/P4) A permutation of the integers $1, 2, \ldots, m$ is called fresh if there exists no positive integer $k < m$ such that the first k numbers in the permutation are $1, 2, \ldots, k$ in some order. Let f_m be the number of fresh permutations of the integers $1, 2, \ldots, m$. Prove that $f_n \geq n \cdot f_{n-1}$ for all $n \geq 3$.

For example, if $m = 4$, then the permutation $(3, 1, 4, 2)$ is fresh, whereas the permutation $(2, 3, 1, 4)$ is not.

4♥ 16. (IMOSL 2002/C2) Let n be an odd positive integer. The unit squares of an $n \times n$ chessboard are coloured alternatively black and white, with four corners coloured black. An L-tromino is an L-shaped formed by three connected unit squares. For which values of n, is it possible to cover all the black squares with non-overlapping L-trominoes? When it is possible, what is the minimum number of L-trominoes needed?



[Hint:]

4♥ 17. Suppose that the edges of the complete graph K_9 is coloured with either red or blue. Suppose that there is no blue triangle. Show that there are four vertices w, x, y and z so that all of the edges wx, wy, wz, xy, xz and yz are red.

4♥ 18. (EGMO 2022/P5) For all positive integers n, k, let $f(n, 2k)$ be the number of ways an $n \times 2k$ board can be fully covered by nk dominoes of size 2×1. (For example, $f(2, 2) = 2$ and $f(3, 2) = 3$.) Find all positive integers n such that for every positive integer k, the number $f(n, 2k)$ is odd.
[Hint:]

4♥ 19. (IMO 2020/P4) There is an integer $n > 1$. There are n^2 stations on a slope of a mountain, all at different altitudes. Each of two cable car companies, A and B, operates k cable cars; each cable car provides a transfer from one of the stations to a higher one (with no intermediate stops). The k cable cars of A have k different starting points and k different finishing points, and a cable car which starts higher also finishes higher. The same conditions hold for B. We say that two stations are linked by a company if one can start from the lower station and reach the higher one by using one or more cars of that company (no other movements between stations are allowed). Determine the smallest positive integer k for which one can guarantee that there are two stations that are linked by both companies.

5♥ 20. (IMOSL 2021/C2) Let $n \geq 3$ be a fixed integer. There are $m \geq n + 1$ beads on a circular necklace. You wish to paint the beads using n colors, such that among any $n + 1$ consecutive beads every color appears at least once. Find the largest value of m for which this task is *not* possible.

5♥ 21. (EGMO 2023/P3) Let k be a positive integer. Lexi has a dictionary $\mathbb{D}$ consisting of some k-letter strings containing only the letters A and B. Lexi would like to write either the letter A or the letter B in each cell of a $k \times k$ grid so that each column contains a string from $\mathbb{D}$ when read from top-to-bottom and each row contains a string from $\mathbb{D}$ when read from left-to-right.

What is the smallest integer m such that if $\mathbb{D}$ contains at least m different strings, then Lexi can fill her grid in this manner, no matter what strings are in $\mathbb{D}$?
[Hint:]

5♥ 22. Three positive real numbers a, b and c are chosen uniformly at random, so that $a + b + c = 1$. What is the probability that a, b and c form the side-lengths of some triangle?

6♥ 23. Prove $e \leq 3$ combinatorially.