Problem Set 4

IMO Training for 2023

24 May, 2023

Problems for this week

This problem set is optional. But you can still collect diamonds in this problem set if you can submit before the next problem solving session begins.

2◆ 1. Total of 2023 real numbers are written on a board. It is found that sum of any 100 of them is positive. Prove that the sum of all 2023 numbers is also positive.

3◆ 2. A positive integer n is called *separable* if we can partition the set $\{n, n+1, n+2, \ldots, n+8\}$ into two sets such that the product of the numbers in each set are equal. Find all separable positive integers.
[Hint:]

4◆ 3. A tennis tournament is played with n people in a round-robin style. Each game is either a win or a loss to a player i.e. there are no draws. At the end of the tournament, let $w_1, w_2, \ldots, w_n$ be the number of wins each person had let $l_1, l_2, \ldots, l_n$ be the number of losses each person had. Prove that

$$w_1^2 + w_2^2 + \cdots + w_n^2 = l_1^2 + l_2^2 + \cdots + l_n^2.$$

[Hint:]

4◆ 4. Call a point in the Cartesian plane with integer coordinates a *lattice point*. Given a finite set $\mathcal{S}$ of lattice points we repeatedly perform the following operation: given two distinct lattice points A, B in $\mathcal{S}$ and two distinct lattice points C, D not in $\mathcal{S}$ such that $ACBD$ is a parallelogram with $AB > CD$, we replace A, B by C, D. Show that only finitely many such operations can be performed.
[Hint:]

5◆ 5. We say that a finite set $\mathcal{S}$ of points in the plane is *balanced* if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is *centre-free* if for any three different points A, B and C in $\mathcal{S}$, there is no point P in $\mathcal{S}$ such that $PA = PB = PC$.

(a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.
[Hint:]

(b) Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.
[Hint:]

For Further Practice

[2♥] 6. (Japan MO/2001) Each square of an $m \times n$ chessboard is painted black or white in such a way that for every black square, the number of black squares adjacent to it is odd (two squares are adjacent if they share one edge). Prove that the number of black squares is even.

[3♥] 7. Show that a graph with n vertices and m edges has at least $\frac{4m}{3n}\left(m - \frac{n^2}{4}\right)$ triangles.

[3♥] 8. Let G be a non-bipartite graph that does not contain a triangle. Show that G has at most $(n-1)^2/4 + 1$ edges.

[4♥] 9. (Canada MO 2018/P1) Consider an arrangement of tokens in the plane, not necessarily at distinct points. We are allowed to apply a sequence of moves of the following kind: select a pair of tokens at points A and B and move both of them to the midpoint of A and B.

We say that an arrangement of n tokens is collapsible if it is possible to end up with all n tokens at the same point after a finite number of moves. Prove that every arrangement of n tokens is collapsible if and only if n is a power of 2.

[4♥] 10. (USAJMO 2020/P1) Let $n \geq 2$ be an integer. Carl has n books arranged on a bookshelf. Each book has a height and a width. No two books have the same height, and no two books have the same width. Initially, the books are arranged in increasing order of height from left to right. In a move, Carl picks any two adjacent books where the left book is wider and shorter than the right book, and swaps their locations. Carl does this repeatedly until no further moves are possible. Prove that regardless of how Carl makes his moves, he must stop after a finite number of moves, and when he does stop, the books are sorted in increasing order of width from left to right.

[4♥] 11. (USAJMO 2019/P1) There are $a+b$ bowls arranged in a row, numbered 1 through $a+b$, where a and b are given positive integers. Initially, each of the first a bowls contains an apple, and each of the last b bowls contains a pear. A legal move consists of moving an apple from bowl i to bowl $i+1$ and a pear from bowl j to bowl $j-1$, provided that the difference $i-j$ is even. We permit multiple fruits in the same bowl at the same time. The goal is to end up with the first b bowls each containing a pear and the last a bowls each containing an apple. Show that this is possible if and only if the product ab is even.

[4♥] 12. (EGMO 2023/P4) Turbo the snail sits on a point on a circle with circumference 1. Given an infinite sequence of positive real numbers $c_1, c_2, c_3, \ldots$, Turbo successively crawls distances $c_1, c_2, c_3, \ldots$ around the circle, each time choosing to crawl either clockwise or counterclockwise.

Determine the largest constant $C > 0$ with the following property: for every sequence of positive real numbers $c_1, c_2, c_3, \ldots$ with $c_i < C$ for all i, Turbo can (after studying the sequence) ensure that there is some point on the circle that it will never visit or crawl across.

[4♥] 13. (USATSTST 2019/P8) Let $\mathcal{S}$ be a set of 16 points in the plane, no three collinear. Let $\chi(S)$ denote the number of ways to draw 8 lines with endpoints in $\mathcal{S}$, such that no two drawn segments intersect, even at endpoints. Find the smallest possible value of $\chi(\mathcal{S})$ across all such $\mathcal{S}$.

[4♥] 14. Let G be a connected graph that does not contain a Hamiltonian cycle. Show that G contains a spanning tree T such that any two of the leaves (vertices of degree 1 in T) are not joined by an edge in G.

[4♥] 15. Suppose that in a certain society, each pair of students can be classified as either *amicable* or *hostile*. We shall say that each member of an amicable pair is a *friend* of the other. Suppose that the society has n people and q amicable pairs, and that for every set of three persons, at least one pair is hostile. Prove that there is at least one member of the society whose foes include $q\left(1-\frac{4q}{n^2}\right)$ or fewer amicable pairs.

5♥ 16. (USATSTST 2016/P5) In the coordinate plane are finitely many walls; which are disjoint line segments, none of which are parallel to either axis. A bulldozer starts at an arbitrary point and moves in the $+x$ direction. Every time it hits a wall, it turns at a right angle to its path, away from the wall, and continues moving. (Thus the bulldozer always moves parallel to the axes.)

Prove that it is impossible for the bulldozer to hit both sides of every wall.

5♥ 17. (Kovari-Sos-Turan Theorem) Let $2 \le s \le t$ be positive integers, and let $K_{s,t}$ denote the complete bipartite graph with partition sets of size s and t. Prove that there is a constant C such that

$$\operatorname{ex}(n, K_{s,t}) \le Cn^{2-\frac{1}{s}}$$

for all positive integers n.

6♥ 18. (IMOSL 2010/C5) $n \ge 4$ players participated in a tennis tournament. Any two players have played exactly one game, and there was no tie game. We call a company four of players *bad* if one player is defeated by one of the players is defeated by the other three players, and these three players formed a cyclic triple (i.e. three players A, B and C such that A beats B, B beats C and C beats A). Suppose that there is no bad company in this tournament. Let w_i and l_i be respectively the number of wins and loses of the i-th player. Prove that

$$\sum_{i=1}^{n}(w_i - l_i)^3 \ge 0.$$