Problem Set 3

IMO Training for 2023

17 May 2023

Problems for this week

Please try to collect at least 5♦'s before the NEXT-NEXT problem solving session.
Note: the above ♦-quota is mandatory for IMO team members.

2♦ 1. In a certain committee, each member belongs to exactly three subcommittees, and each subcommittee has exactly three members. Prove that the number of members equals to the number of subcommittees.
[Hint:]

2♦ 2. Let G be a graph whose minimum degree is δ. Show that G contains a path with $\delta + 1$ vertices. (If you don't understand the technical terms in this question, try searching online or wait for my graph-terms handout to be posted in google classroom)
[Hint:]

3♦ 3. There are $n \geq 2$ students in a school some of whose are friends with each other. Friendship is mutual i.e. if A is friends with B, then B is also friends with A. The class teacher is going to separate them into two rooms. A student will be *sad* if at least half of his friends are in the room different from his. Show that the teacher can make every student sad.
[Hint:]

3♦ 4. Suppose 5000 distinct points in the plane are given such that no four points are collinear. Show that it is possible to select 100 of the points for which no three points are collinear.
[Hint:]

4♦ 5. (IMO 1989/P3) Let n and k be positive integers and let $\mathcal{S}$ be a set of n points in the plane such that

- no three points of $\mathcal{S}$ are collinear, and
- for every point P of $\mathcal{S}$, there are at least k points equidistant from P.

Prove that $k < \frac{1}{2} + \sqrt{2n}$.
[Hint:]

For Further Practice

4♥ 6. (IMO 1998/P2) In a contest, there are m candidates and n judges, where $n \geq 3$ is an odd integer. Each candidate is evaluated by each judge as either pass or fail. Suppose that each pair of judges agrees on at most k candidates. Prove that

$$\frac{k}{m} \geq \frac{n-1}{2n}.$$

4♥ 7. (IMOSL 2013/C1) Let n be an positive integer. Find the smallest integer k with the following property: given any real numbers $a_1, \dots, a_d$ such that $a_1 + a_2 + \cdots + a_d = n$ and $0 \leq a_i \leq 1$ for $i = 1, 2, \dots, d$, it is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.

4♥ 8. Show that a graph with average degree at least $2k$ contains a subgraph with minimum degree at least k.
[Hint:]

4♥ 9. (ELMO SL 2011/C2) A directed graph has each vertex with outdegree 2. Prove that it is possible to split the vertices into 3 sets so that for each vertex v, v is not simultaneously in the same set with both of the vertices it points to.
[Hint:]

4♥ 10. (Swiss TST 2018/P5) Let n be a positive integer. We consider $n \times n$ grid. We colour k squares in black, such that given any three columns, there exists a most one row that intersects the three columns at a black square.

$$k \leq \frac{n\sqrt{8n-7}+n}{2}.$$

[Hint:]

4♥ 11. (Zackendorf's Theorem) Show that every positive integer can be uniquely written as the sum of non-consecutive Fibonacci numbers.
[Hint:]

4♥ 12. (IMOSL 2016/C3) Let n be a positive integer relatively prime to 6. We paint the vertices of a regular n-gon with three colours so that there is an odd number of vertices of each colour. Show that there exists an isosceles triangle whose three vertices are of different colours.

4♥ 13. (IMO 2003/P1) Let A be a 101-element subset of the set $S = \{1, 2, \dots, 1000000\}$. Prove that there exist numbers $t_1, t_2, \dots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \qquad j = 1, 2, \dots, 100$$

are pairwise disjoint.

5♥ 14. (IMO 1989/P3 Harder) Let n and k be positive integers and let $\mathcal{S}$ be set of n points in the plane such that for every point P of $\mathcal{S}$, there are at least k points of $\mathcal{S}$ equidistant from P. Prove that $k < \frac{1}{2} + \sqrt{2n}$.

5♥ 15. (Croatia TST 2011/P2) There are $n \geq 4$ people at a party among whom some are friends (friendship is mutual). Among any 4 of them, there are either 3 who are all friends with each other or 3 who aren't friends with each other. Prove that the people can be separated into two groups A and B such that everyone in A knows each other and everyone in B does not know each other.

5♥ 16. Jamie plays a game with a hexagon. He writes an integer on each vertex of the hexagon such that the sum of 6 numbers is 2013. Each step he will choose a number on a vertex and replaces with the non-negative difference of the numbers on 2 adjacent vertices. Prove that Jamie can make all the numbers equal to zero in a finite number of steps.

5♥ 17. (Italy TST 1993/P4) An $m \times n$ chessboard with $m, n \geq 2$ given. Some dominoes are placed on the chessboard so that the following conditions are satisfied:

- Each domino occupies two adjacent squares of the chessboard,
- it is not possible to put another domino onto the chessboard without overlapping,
- it is not possible to slide a domino horizontally or vertically without overlapping.

Prove that the number of squares that are not covered by a domino is less than $\frac{1}{5}mn$.
[Hint:]

6♥ 18. (IMO 2014/P5) For each positive integer n, the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.
[Hint:
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6♥ 19. (IMO 2016/P2) Find all integers n for which each cell of $n \times n$ table can be filled with one of the letters I, M and O in such a way that:

- in each row and each column, one third of the entries are I, one third are M and one third are O; and
- in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are I, one third are M and one third are O.

Note: The rows and columns of an $n \times n$ table are each labelled 1 to n in a natural order. Thus each cell corresponds to a pair of positive integer (i, j) with $1 \leq i, j \leq n$. For $n > 1$, the table has $4n - 2$ diagonals of two types. A diagonal of first type consists all cells (i, j) for which $i + j$ is a constant, and the diagonal of this second type consists all cells (i, j) for which $i - j$ is constant.

6♥ 20. Show that every connected graph G with n vertices and minimum degree δ contains a path of length $\min\{2\delta, n - 1\}$.
[Hint:]
Bonus: 2♥ Prove Dirac's theorem (link): if every vertex has degree at least $n/2$, then G contains a cycle passing through every vertex.