Problem Set 2

IMO Training for 2023

10 May 2023

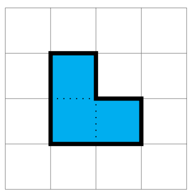
Problems for this week

Please try to collect at least 5♦'s before the next problem solving session.
Note: the above ♦-quota is mandatory for IMO team members.

2♦ 1. There are three piles with n tokens each. In every step we are allowed to choose two piles, take one token from each of those two piles and add a token to the third pile. Using these moves, is it possible to end up having only one token?
[Hint:]

2♦ 2. Cockroach Joey is sitting at the lower left corner of an $n \times n$ grid. Each second, Joey crawls to the adjacent square (two squares are adjacent if they share a side). Its goal is to to crawl through every square of the grid exactly once and end up at the upper right corner. Find all possible values of n for which Joey can achieve its goal.
[Hint:]

3♦ 3. One of the 1×1 cells of a given $2^n \times 2^n$ grid is removed. Show that it is possible to tile the resulting grid using L-shaped tiles shown below.



You may rotate or reflect the tiles, the tiles shall not exceed the boundaries of the grid and may not overlap.
[Hint:]

4♦ 4. Alice and Barbara play a game with a pack of $2n$ cards, on each of which is written a positive integer. The pack is shuffled and the cards laid out in a row, with the numbers facing upwards. Alice starts, and the girls take turns to remove one card from either end of the row, until Barbara picks up the final card. Each girl's score is the sum of the numbers on her chosen cards at the end of the game. Prove that Alice can always obtain a score at least as great as Barbara's.
[Hint:]

4♦ 5. Let S be a set with 2002 elements, and let N be an integer with $0 \leq N \leq 2^{2002}$. prove that it is possible to color every subset of S either black or white so that the following conditions hold:

- the union of any two white subsets is white,
- the union of any two black subsets is black,
- there are exactly N white subsets.

[Hint:]

4♦ 6. Prove that we can rearrange the terms of the sequence $1, 2, 3, \ldots, 2023$ so that the average of any two distinct terms does not appear between them in the rearranged sequence.

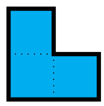
[Hint:]

For Further Practice

2♥ 7. Euclidean plane is divided into many regions by a finite number of circles. Show that it is possible to colour the regions red or blue so that no two regions of the same colour share a border of non-zero length.
[Hint:]

3♥ 8. Suppose we are given 1000 lamps and 1000 switches with each switch connected to one lamp and each lamp is connected to one switch, but we do not know which lamp corresponds to which switch. Initially, all lamps are off. In one operation, we can specify an arbitrary set of switches, and all of them will be switched from off to on simultaneously. We will then see which lamps come on, then we switch all the lights back off. Show that we can know which switch corresponds to which lamp within 10 operations.
[Hint:]

4♥ 9. (Cyberspace MO 2020/P1) Consider an $n \times n$ unit-square board. The main diagonal of the board is the n unit squares along the diagonal from the top left to the bottom right. We have an unlimited supply of tiles of this form:



The tiles may be rotated. We wish to place tiles on the board such that each tile covers exactly three unit squares, the tiles do not overlap, every unit square on the main diagonal is not covered, and all other unit squares are covered exactly once. For which $n \geq 2$ is this possible?

4♥ 10. Let n be a positive integer. Show that there exists a subset A of $\{1, 2, \ldots, 3^n\}$ such that $|A| = 2^n$ and A does not contain a non-trivial 3-term arithmetic progression i.e. if $x, y, z \in A$ are distinct, then $z - y \neq y - x$.
[Hint:]

4♥ 11. (IMOSL 2015/C1) In Lineland there are $n \geq 1$ towns, arranged along a road running from left to right. Each town has a left bulldozer (put to the left of the town and facing left) and a right bulldozer (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a left and right bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let A and B be two towns, with B to the right of A. We say that town A can sweep town B away if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly town B can sweep town A away if the left bulldozer of B can move over to A pushing off all bulldozers of all towns on its way.

Prove that there is exactly one town that cannot be swept away by any other one.
[Hint:]

4♥ 12. (IMOSL 2009/C1) Consider 2009 cards, each having one gold side and one black side, lying parallel on a long table. Initially all cards show their gold sides. Two players, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turining them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins.

(a) Does the game necessarily end?

(b) Does there exist a winning strategy for the starting player?

4♥ 13. Let G be a graph without an odd cycle i.e. for any collection of odd number of (distinct) vertices $v_1, \ldots, v_{2k-1}$ of G, at least one of the edges

$$v_1v_2, v_2v_3, \ldots, v_{2k-2}v_{2k-1}, v_{2k-1}v_1$$

is absent. Show that it is possible to colour the vertices either black or white so that no two vertices of the same colour are joined by an edge.

5♥ 14. (EGMO 2017/3) There are 2017 lines in the plane such that no three of them go through the same point. Turbo the snail sits on a point on exactly one of the lines and starts sliding along the lines in the following fashion: she moves on a given line until she reaches an intersection of two lines. At the intersection, she follows her journey on the other line turning left or right, alternating her choice at each intersection point she reaches. She can only change direction at an intersection point. Can there exist a line segment through which she passes in both directions during her journey?
[Hint:]

6♥ 15. (GQMO 2020/S7) Each integer in $\{1, 2, 3, ..., 2020\}$ is coloured in such a way that, for all positive integers a and b such that $a + b \leq 2020$, the numbers a, b and $a + b$ are not coloured with three different colours. Determine the maximum number of colours that can be used.

7♥ 16. (IMO 2017/5) Let $N \geq 2$ be an integer. $N(N+1)$ soccer players, no two of the same height, stand in a row in some order. Coach Ralph wants to remove $N(N-1)$ people from this row so that in the remaining row of $2N$ players, no one stands between the two tallest ones, no one stands between the third and the fourth tallest ones, $\ldots$, and finally no one stands between the two shortest ones. Show that this is always possible.
[Hint:]

7♥ 17. (IMOSL 2005/C5) There are n markers, each with one side white and the other side black. In the beginning, these n markers are aligned in a row so that their white sides are all up. In each step, if possible, we choose a marker whose white side is up (but not one of the outermost markers), remove it, and reverse the closest marker to the left of it and also reverse the closest marker to the right of it. Prove that, by a finite sequence of such steps, one can achieve a state with only two markers remaining if and only if $n - 1$ is not divisible by 3.
[Hint:]