Problem Set 1

IMO Training for 2023

10 May 2023

Problems for this week

Please try to collect at least [5♦]'s before the next problem solving session.
Note: the above [♦]-quota is mandatory for IMO team members.

[2♦] 1. Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Three frogs Alice, Bob and Chris are sitting together at a point on the Euclidean plane. For each $1 \leq k \leq n$, at the k-th round, each of them jump a_k units in the direction of their own choice: north or east. Show that after all the rounds, the positions of Alice, Bob and Chris are collinear.

[2♦] 2. In an online shooter game, 100 players participate in a match. Once the game starts, players can start to shoot and kill each other. Players come back to life after death (that is, a player can be killed more than once) and they cannot kill themselves. The *score* of each player starts with 1, and gets $+1$ for each kill and -2 for each death (negative scores are possible). At the end of the match, it was found that the sum of the scores of all the players is 0. Prove that no player got more than 100 kills.

[2♦] 3. 2023 cups are placed upside down on a table. You can take two cups at a time and flip them. Is it possible to make every cup right-side up?

[3♦] 4. There are $2n + 1$ lamps placed in a circle. Each day, some of the lamps change state (from on to off or off to on), according to the following rules. On the k-th day, if a lamp is in the same state as at least one of its neighbors, then it will not change state the next day. If a lamp is in a different state from both of its neighbors on the k-th day, then it will change its state the next day. Show that regardless of the initial states of each lamp, after some point none of the lamps will change state.

[3♦] 5. A non-zero real number is written in each cell of an $m \times n$ table. You are allowed to pick any row or column, and change the signs of every number lying in it. Show that it is possible make a sequence of moves so that sum of the entries in each row or column is non-negative.

For Further Practice

3♥ 6. Consider an 8×8 chessboard with usual colouring (i.e. alternating pattern of black and white). In each *move* you are allowed to change the colour of all squares in a row, or in a column, or in a 2×2 square. Is it possible to do a sequence of moves so that we have only one black square in the end?

3♥ 7. (From lecture 2) A game of pebbles is going to be played in the first quadrant of the coordinate plane. Initially, there is one pebble at the origin. In a move, Botez can remove a pebble from (i, j) and place one pebble each on $(i+1, j)$ and $(i, j+1)$, provided that (i, j) had a pebble to begin with and that $(i+1, j)$ and $(i, j+1)$ did not have pebbles. Prove that at any point in the game, there will be a pebble below or on the line $x + y = 2$.

3♥ 8. The points $A_1, \ldots, A_{2n}$ and M lie on the circle. The points $A_1, \ldots, A_{2n}$ are divided into pairs and a straight line is drawn through each pair. Prove that the product of the distances form M to all n drawn lines does not depend on the division of points into pairs.

4♥ 9. (Rearrangement Inequality) Let $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$ be real numbers. Suppose that $y'_1, \ldots, y'_n$ is the permutation of the sequence $y_1, \ldots, y_n$. Show that

$$x_1y_1 + x_2y_2 + \cdots + x_ny_n \geq x_1y'_1 + x_2y'_2 + \cdots + x_ny'_n \geq x_1y_n + x_2y_{n-1} + \cdots + x_ny_1.$$

4♥ 10. (GQMO 2020/B2) The Bank of Zürich issues coins with an H on one side and a T on the other side. Alice has n of these coins arranged in a line from left to right. She repeatedly performs the following operation: if some coin is showing its H side, Alice chooses a group of consecutive coins (this group must contain at least one coin) and flips all of them; otherwise, all coins show T and Alice stops. For instance, if $n = 3$, Alice may perform the following operations:

$$THT \to HTH \to HHH \to TTH \to TTT.$$

She might also choose to perform the operation $THT \to TTT$. For each initial configuration C, let $m(C)$ be the minimal number of operations that Alice must perform. For example, $m(THT) = 1$ and $m(TTT) = 0$. For every integer $n \geq 1$, determine the largest value of $m(C)$ over all 2^n possible initial configurations C.

4♥ 11. (IMOSL 2012/C1) Several positive integers are written in a row. Iteratively, Alice chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y + 1, x)$ or $(x - 1, x)$. Prove that she can perform only finitely many such iterations.

4♥ 12. (Russia 1997) There are some stones placed on a infinite (in both directions) row of squares labeled by integers. There may be more than one stone on each square. There are two types of moves:

- remove one some from each of the squares n and $n - 1$, and place one stone on $n + 1$,
- remove two stones from square n and place one stone on each of the squares $n + 1$ and $n - 2$.

Show that at some point, no more moves can be made.
Bonus: 5♥ Show that the ending configuration is independent of the choices of moves.

4♥ 13. (USAMO 2017/P4) Let $P_1, P_2, \ldots, P_{2n}$ be $2n$ distinct points on the unit circle $x^2 + y^2 = 1$, other than $(1, 0)$. Each point is colored either red or blue, with exactly n red points and n blue points. Let $R_1, R_2, \ldots, R_n$ be any ordering of the red points. Let B_1 be the nearest blue point to R_1 traveling counterclockwise around the circle starting from R_1. Then let B_2 be the nearest of the remaining blue points to R_2 travelling counterclockwise around the circle from R_2, and so on, until we have labeled all of the blue points $B_1, \ldots, B_n$. Show that the number of counterclockwise arcs of the form $R_i \to B_i$ that contain the point $(1,0)$ is independent of the way we chose the ordering $R_1, \ldots, R_n$ of the red points.

4♥ 14. (All-Russian MO 1994) On a line are given n blue and n red points. Prove that the sum of distances between pairs of points of the same color does not exceed the sum of distances between pairs of points of different colors.

5♥ 15. (IMO 2019/P5) The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are $k > 0$ coins showing H, then he turns over the k-th coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$, the process starting with the configuration THT would be

$$THT \to HHT \to HTT \to TTT,$$

which stops after three operations.

(a) Show that for each initial configuration, Harry stops after a finite number of operations.

(b) For each initial configuration C, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C.