Problem 2. Some cells of an infinite chessboard (infinite in all directions) are coloured *blue* so that at least one of the 100 cells in any 10×10 rectangular grid is blue. Prove that, for any positive integer n, it is possible to select n rows and n columns so that all of the n^2 cells in their intersections are blue.

Solution. Let $N = 10(10n+1)$. Note that any $N \times 10$ rectangular grid can be tiled with $10n+1$ grids of size 10×10. Hence, any $N \times 10$ rectangular contains $10n+1$ blue cells and thus, there is a column containing n blue cells by pigeonhole principle.

Now, let $M = 10\left(\binom{N}{n}n+1\right)$ and consider a $N \times M$ rectangular grid $\mathcal{G}$. Note that $\mathcal{G}$ can be tiled with $\binom{N}{n}n+1$ subgrids of size $N \times 10$. Each of these subgrids contains a column with at least n blue cells out of N. Thus, by again pigeonhole principle, there are n columns and n rows in $\mathcal{G}$, all of whose intersections are blue. □

Comment: This problem can be nuked with Kóvári-Sős-Turán theorem [1] as follows. For each positive integer N, pick an arbitrary $10N \times 10N$ rectangular grid and consider the bipartite graph G_N whose vertices are rows and columns with edges between those rows and columns that intersect at a blue square. Then G_N contains $\Omega(N^2)$ edges. Since $\mathrm{ex}(N, K_{n,n}) = o(N^2)$ by Kóvári-Sős-Turán, G_N contains $K_{n,n}$ as a subgraph for sufficiently large N.

[1] `https://mathoverflow.net/questions/297118/kovari-sos-turan-theorem#:~:text=Let%20r%E2%89%A4s%20be,s%20for%20a%20constant%20c.`

Marking Scheme for Problem 2

- (2 points) Showing that for a suffiently large N depending only on n, the grid of size $10 \times N$, contains a row with n blue cells.
- (7 points) For completing a full solution, possibly with minor mistakes.
- For the write-ups for which it is suspectable that the student managed to grasp the main idea of the problem, point-deduction will be made for each flaw that is described below:
  - (-1 point) Confusing write-ups that can cause more than one reasonable non-trivial meaning.
  - (-1 point) Only considering the worst case instead of using pigeonhole.
- No points will be deducted for the following minor flaws:
  - Incorrect grammar and spelling,
  - Some minor calculation mistakes here and there.

Problem 7. Let $n \geq 2$ be a positive integer. Total of $2n$ balls are coloured with n colours so that there are two balls of each colour. These balls are put inside n cylinderical boxes with two balls in each box, one on top of the other.

First, Phoe Wa Lone looks at the boxes and reverses the (top/bottom) order of the balls in the boxes of his choice. Then, he takes an empty cylinderical box and starts playing the sorting game as follows. In a *move*, he can take a ball b at the top of any non-empty box and either put it inside an empty box or put it in the box only containing the ball of the same colour as b. His goal is to sort the balls so that balls of the same colour are grouped together in each box.

Find the smallest positive integer N (in terms of n) such that no matter what initial placement of balls is given, Phoe Wa Lone can achieve his goal using at most N moves.

Solution. The answer is $N = n + \lfloor n/2 \rfloor$. We will first show that any initial configuration can be solved within $n + \lfloor n/2 \rfloor$ moves. Without loss of generality, we may assume that no box contains two balls of the same colour since Phoe Wa Lone can just ignore those boxes and work with the remaining without affecting the minimum number of moves required.

Claim 1. *Phoe Wa Lone can always reverse the orders before starting the game so that all n balls at the top of each box have different colours.*

Proof. Call a sequence $B_1, B_2, \ldots, B_k$ of boxes a *chain* if B_i and B_{i+1} share a ball of the same colour for each $i = 1, \ldots, k-1$, and B_k and B_1 also share the same colour. We will show that Phoe Wa Lone can reduce the number of pairs p of boxes whose top balls have the same colour. If $p > 0$, without loss of generality, we may suppose that the top balls of B_1 and B_k are the same. Then, since here are only $k-1$ colours present in the bottom balls, there are two adjacent boxes B_i and B_{i+1} $(i = 1, 2, \ldots, k-1)$ whose bottom balls have the same colour. Let i be the smallest with this property. Now, Phoe Wa Lone shall reverse the order of the boxes $B_1, B_2, B_3, \ldots, B_i$. Note that this reduces the value of p. ■

After reversing the boxes as in claim 1, each chain $B_1, B_2, \ldots, B_k$ satisfies the property: bottom ball of B_i and top ball of B_{i+1} have the same colour for $i = 1, 2, \ldots, k-1$, and the bottom ball of B_k and the top ball of B_1 have the same colour. We will call such a chain *sorted*. Note that at this point, the boxes can be partitioned into sorted chains, each containing at least 2 boxes. Thus, there are at most $\lfloor n/2 \rfloor$ sorted chains.

Claim 2. *The minimum number of moves needed to achieve Phoe Wa Lone's goal on each sorted chain $B_1, B_2, \ldots, B_k$ is equal to $k+1$.*

Proof. Let E be the empty box. Note that applying following $k+1$ moves lets Phoe Wa Lone achieve his goal:

$$B_1 \to E, \quad B_2 \to B_1, \quad B_3 \to B_2, \quad \cdots \quad B_k \to B_{k-1}, \quad E \to B_k.$$

Note that this set of moves leaves E empty at the end. To see that $k+1$ moves are necessary, note that we need to spend one move each to make the balls in each B_i $(i = 1, \ldots, k)$ the same since none of them contain same coloured balls. The extra one moves comes from the fact that the first move must be used to put a ball in the empty box. ■

Applying claim 2 on each sorted chain shows that $n + \lfloor n/2 \rfloor$ moves suffice. For an example that needs at least $n + \lfloor n/2 \rfloor$, we can consider the following: Let $c_1, c_2, \ldots, c_n$ be the colours.

- For n even, consider the boxes $(c_1, c_2), (c_1, c_2), (c_3, c_4), (c_3, c_4), \ldots, (c_{n-1}, c_n), (c_{n-1}, c_n)$.
- For n odd, consider the boxes $(c_1, c_2), (c_2, c_3), (c_3, c_1), (c_4, c_5), (c_4, c_5), \ldots, (c_{n-1}, c_n), (c_{n-1}, c_n)$.

Note that the argument in claim 2 showing that $k+1$ moves are necessary applies to all chains that are not necessarily sorted. Thus, the initial configurations shown above need at least $n + \lfloor n/2 \rfloor$ moves to solve. □

Comment 1. In fact, the provided solution gives the minimum number of moves $m(\mathcal{C})$ to any initial configuration $\mathcal{C}$. Let $u(\mathcal{C})$ be the number of boxes that contain differently coloured balls, and let $v(\mathcal{C})$ be the number of chains with at least 2 boxes. Then,

$$m(\mathcal{C}) = u(\mathcal{C}) + v(\mathcal{C}).$$

Comment 2. If Phoe Wa Lone is not allowed to reverse the order of the balls, with a little bit more effort, we can characterize the set of all possible initial configurations:

Phoe Wa Lone can achieve his goal with initial configuration $\mathcal{C}$ if and only if every chain in $\mathcal{C}$ contains exactly one or no pairs of boxes whose top balls have the same colour.

The original version of the problem requires one to find this characterization without explicitly asking it. But, the problem was changed because this version might be too difficult for the mock IMO.

Comment 3. In the setting where Phoe Wa Lone cannot alter the initial configuration, the problem of deciding whether or not an initial configuration can be solved for a general setting: kn balls with k balls of each colour, has been shown to be NP-complete[2].

[2] Ito, Takehiro, et al. *Sorting Balls and Water: Equivalence and Computational Complexity.* arXiv preprint arXiv:2202.09495 (2022).

Marking Scheme for Problem 7

- (+2 points) *Lower-bound construction.* For correctly proving that $N \geq n + \lfloor n/2 \rfloor$.
  - (+1 point) Describing the correct constructions.
  - (+1 point) Proving that $n + \lfloor n/2 \rfloor$ moves are required for those constructions.
  - (−1 point) Forgetting to consider more than one case during the proof of the correctness (only applicable if the student's solution uses case bash).
- (+5 points) *Upper-bound construction.* For correctly proving that $N \leq n + \lfloor n/2 \rfloor$.
  - (+2 points) Showing that Phoe Wa Lone can achieve his goal on chains.
  - (+2 points) Proving claim 1. The weaker version of leaving at most one pair of same coloured balls at the top for each chain is also acceptable.
  - (+1 point) Correctly counting the number of moves used.
  - (−1 point) Using "repeating until stuck" method without showing that the process ends or without showing that the process loops *to the beginning*.
- For the write-ups for which it is suspectable that the student managed to grasp the main idea of either part of the problem, point-deduction will be made for each flaw that is described below:
  - (−1 point) Confusing write-ups that can cause more than one reasonable non-trivial meaning.
  - (−1 point) Claiming and proving the wrong value of N (for example $n + \lceil n/2 \rceil$) resulting from correct constructions with incorrect calculations.
  - (−1 point) Claiming the correct value of N, but not writing it in a closed form.
- No points will be deducted for the following minor flaws:
  - Incorrect grammar and spelling,
  - Some minor calculation mistakes here and there,
  - Accidently writing $3n/2$ here and there instead of $n + \lfloor n/2 \rfloor$.