Problem 2. Some cells of an infinite chessboard (infinite in all directions) are coloured *blue* so that at least one of the 100 cells in any 10×10 rectangular grid is blue. Prove that, for any positive integer n, it is possible to select n rows and n columns so that all of the n^2 cells in their intersections are blue.

Solution. Let $N = 10(10n+1)$. Note that any $N \times 10$ rectangular grid can be tiled with $10n+1$ grids of size 10×10. Hence, any $N \times 10$ rectangular contains $10n+1$ blue cells and thus, there is a column containing n blue cells by pigeonhole principle.

Now, let $M = 10\left(\binom{N}{n}n+1\right)$ and consider a $N \times M$ rectangular grid $\mathcal{G}$. Note that $\mathcal{G}$ can be tiled with $\binom{N}{n}n+1$ subgrids of size $N \times 10$. Each of these subgrids contains a column with at least n blue cells out of N. Thus, by again pigeonhole principle, there are n columns and n rows in $\mathcal{G}$, all of whose intersections are blue. $\square$

Comment: This problem can be nuked with Kóvári-Sős-Turán theorem [1] as follows. For each positive integer N, pick an arbitrary $10N \times 10N$ rectangular grid and consider the bipartite graph G_N whose vertices are rows and columns with edges between those rows and columns that intersect at a blue square. Then G_N contains $\Omega(N^2)$ edges. Since $\mathrm{ex}(N, K_{n,n}) = o(N^2)$ by Kóvári-Sős-Turán, G_N contains $K_{n,n}$ as a subgraph for sufficiently large N.

[1] `https://mathoverflow.net/questions/297118/kovari-sos-turan-theorem#:~:text=Let%20r%E2%89%A4s%20be,s%20for%20a%20constant%20c.`

Marking Scheme for Problem 2

- (2 points) Showing that for a suffiently large N depending only on n, the grid of size $N \times 10$, contains a column with n blue cells.
- (7 points) For completing a full solution, possibly with minor mistakes.
  - (-3 points) For working on a possibly sparse set of rows instead of N consecutive rows in the second part of the solution.
- For the write-ups for which it is suspectable that the student managed to grasp the main idea of the problem, point-deduction will be made for each flaw that is described below:
  - (-1 point) Confusing write-ups that can cause more than one reasonable non-trivial meaning.
  - (-1 point) Only considering the worst case instead of using pigeonhole.
- No points will be deducted for the following minor flaws:
  - Incorrect grammar and spelling,
  - Some minor calculation mistakes here and there.

Problem 7. Let $n \geq 2$ be a positive integer. Total of $2n$ balls are coloured with n colours so that there are two balls of each colour. These balls are put inside n cylinderical boxes with two balls in each box, one on top of the other. Phoe Wa Lone has an empty cylinderical box and his goal is to sort the balls so that balls of the same colour are grouped together in each box. In a *move*, Phoe Wa Lone can do one of the followings:

- Select a box with exactly two balls and reverse the order of top and bottom balls.
- Take a ball b at the top of a non-empty box and either put it inside a empty box or put it in the box only containing the ball of the same colour as b.

Find the smallest positive integer N such that for any initial placement of the balls, Phoe Wa Lone can always achieve his goal using at most N moves in total.

Solution. The answer is $N = n + \lfloor n/2 \rfloor$. We will first show that any initial configuration can be solved within $n + \lfloor n/2 \rfloor$ moves. Without loss of generality, we may assume that no box contains two balls of the same colour since Phoe Wa Lone can just ignore those boxes and work with the remaining without affecting the minimum number of moves required. Call a sequence $B_1, B_2, \ldots, B_k$ of boxes a *chain* if B_i and B_{i+1} share a ball of the same colour for each $i = 1, \ldots, k-1$, and B_k and B_1 also share the same colour. Note that the boxes can be partitioned into disjoint union of chains. Phoe Wa Lone's strategy is to achieve his goal on each chain.

Claim 1. *Suppose $B_1, B_2, \ldots, B_k$ is a chain in which at most two balls of the same colour lie at the top of the respective boxes. Then, minimum number of moves that Phoe Wa Lone needs to achieve his goal on this chain is equal to $k+1$.*

Proof. We will first show that $k+1$ moves is necessary. Indeed, since no box contain two balls of the same colour, at least k moves will be needed to sort each colour appearing in the chain, one move for each colour. Also, the very first non-reversal move must involve the empty box. Hence, he needs at least $k+1$ moves.

Now, we show that $k+1$ moves suffice. Suppose first that no two top balls have the same colour. Without loss of generality, suppose that the bottom ball of B_1 and the top ball of B_2 have the same colour. Then, we can perform the following sequence of $k+1$ moves:

$$B_1 \to E, \quad B_2 \to B_1, \quad B_3 \to B_2, \quad \cdots \quad B_k \to B_{k-1}, \quad E \to B_k$$

where E is the empty box. Now, suppose without loss of generality that the top balls of B_1 and B_k have the same colour, and all other top balls have distinct colours. Since only $k-1$ colours are present among the bottom balls, there are two adjacent boxes B_i and B_{i+1} whose bottom balls have the same colour. Then, we can perform the following sequence of $k+1$ moves:

$$B_1 \to E, \quad B_k \to E,$$

$$B_2 \to B_1, \quad B_3 \to B_2, \quad \cdots \quad B_i \to B_{i-1},$$

$$B_{k-1} \to B_k, \quad B_{k-2} \to B_{k-1}, \quad \cdots \quad B_{i+1} \to B_{i+2}, \quad B_i \to B_{i+1}$$

where E is the empty box. Note that in any case, one of the boxes is left empty. ■

Call a chain satisfying the property in the hypothesis of claim 1 *nice*. Now, Phoe Wa Lone's task is to make all the chains nice.

Claim 2. *Suppose $B_1, B_2, \ldots, B_k$ is a chain. Then, Phoe Wa Lone can make this chain nice using at most $\lfloor k/2 \rfloor - 1$ moves.*

Proof. Suppose that the chain is not nice in the first place. Then, we can assume without loss of generality that the top balls of B_1 and B_k have the same colour. Now, Phoe Wa Lone shall paint the boxes $B_1, \ldots, B_k$ in red or blue as follows: start by painting B_1 with red. When box B_i is painted, he shall colour box B_{i+1} the opposite colour as B_i if either their top balls or bottom balls have the same colour. Otherwise, he colours B_{i+1} with the same colour as B_i. Note that reversing top/bottom order of the balls on all red boxes, or all blue boxes will make the colours of all top balls different. So, the chain will be nice if Phoe Wa Lone flips all the boxes painted with the same colour, except one.

Now, note by pigeonhole principle that at most $\lfloor k/2 \rfloor$ boxes will be red, or at most $\lfloor k/2 \rfloor$ will be blue. So, Phoe Wa Lone can make the chain nice in $\lfloor k/2 \rfloor - 1$ moves. ■

Now Phoe Wa Lone applies the claim 2 first, followed by claim 1 to each chain. On a chain consisting of k boxes, this uses at most

$$(k+1) + \left(\left\lfloor \frac{k}{2} \right\rfloor - 1 \right) = k + \left\lfloor \frac{k}{2} \right\rfloor$$

moves. Adding up over all chains, we can see that Phoe Wa Lone can achieve his goal within $n + \lfloor n/2 \rfloor$ moves.

Now, we will describe an initial configuration which requires at least $n + \lfloor n/2 \rfloor$ moves to achieve Phoe Wa Lone's goal. Let $c_1, c_2, \ldots, c_n$ be the colours.

- For n even, consider the boxes $(c_1, c_2), (c_1, c_2), (c_3, c_4), (c_3, c_4), \ldots, (c_{n-1}, c_n), (c_{n-1}, c_n)$.
- For n odd, consider the boxes $(c_1, c_2), (c_2, c_3), (c_3, c_1), (c_4, c_5), (c_4, c_5), \ldots, (c_{n-1}, c_n), (c_{n-1}, c_n)$.

It follows from claim 1 that Phoe Wa Lone needs at least $n + \lfloor n/2 \rfloor$ moves to solve. □

Comment 1. The initial configurations that need at least $n + \lfloor n/2 \rfloor$ moves are not unique. For example, the following example also works although it is more difficult to prove:

- For n even, consider the boxes

$$(c_1, c_n), (c_1, c_2), (c_3, c_2), (c_3, c_4), (c_5, c_4), \ldots, (c_{n-1}, c_{n-2}), (c_{n-1}, c_n).$$

- For n odd, consider the boxes

$$(c_1, c_n), (c_1, c_2), (c_3, c_2), (c_3, c_4), (c_5, c_4), \ldots, (c_{n-4}, c_{n-3}), (c_{n-2}, c_{n-3}), (c_{n-2}, c_{n-1}), (c_{n-1}, c_n).$$

Comment 2. Phoe Wa Lone can always achieve his goal without reversing the orders of the balls if he is initially given two empty boxes.

Comment 3. In the setting where Phoe Wa Lone is not allowed to change the order of the balls in each box, the problem of deciding whether or not an initial configuration can be solved in a general setting: kn balls with k balls of each colour, has been shown to be NP-complete[2].

[2] Ito, Takehiro, et al. *Sorting Balls and Water: Equivalence and Computational Complexity.* arXiv preprint arXiv:2202.09495 (2022).

Marking Scheme for Problem 7

- (+2 points) *Lower-bound construction.* For correctly proving that $N \geq n + \lfloor n/2 \rfloor$.
  - (+1 point) Describing the correct constructions possibly with an incorrect claim on the number of moves necessary.
  - (+1 point) Proving that $n + \lfloor n/2 \rfloor$ moves are required for those constructions.
  - (-1 point) Forgetting to consider more than one case during the proof of the correctness (only applicable if the student's solution uses case bash).
- (+5 points) *Upper-bound construction.* For correctly proving that $N \leq n + \lfloor n/2 \rfloor$. In the following
  - (+1 point) For partitioning the boxes into disjoint chains.
  - (+1 point) Proving the sufficiency part of claim 1 possibly with no/wrong consideration on the number of moves.
  - (+1 point) Correctly counting the number of moves spent on each chain in claim 1.
  - (+1 point) Proving claim 2 possibly with no/wrong consideration on the number of moves.
  - (+1 point) Correctly counting the number of moves spent on each chain in claim 2.
- Suppose that the student's solution to the sufficiency part achieves $N < \infty$ but does not follow the strategy described in the official solution. In this situation, 2 or 1 points will be awarded depending on whether $N = O(n)$ or not, even if the number of moves is not explicitly counted.
- For the write-ups for which it is suspectable that the student managed to grasp the main idea of either part of the problem, point-deduction will be made for each flaw that is described below:
  - (-1 point) Describing the idea only in the worst case.
  - (-1 point) Confusing write-ups that can cause more than one reasonable non-trivial meaning.
  - (-1 point) Claiming and proving the wrong value of N (for example $n + \lceil n/2 \rceil$) resulting from correct constructions with incorrect calculations.
  - (-1 point) Claiming the correct value of N, but not writing it in a closed form.
- No points will be deducted for the following minor flaws:
  - Incorrect grammar and spelling,
  - Some minor calculation mistakes here and there,
  - Accidently writing $3n/2$ here and there instead of $n + \lfloor n/2 \rfloor$.