Some Housekeeping

- Diamond problems for Problem Set 6 will be due tomorrow. Note that Problem Set 7 will be optional due to mock IMO.

- Which schedule do you prefer for problem solving session?
  - Schedule 1: Regular schedule: Problem solving session for this week's techniques
  - Schedule 2: Help you guys revise and continue with this week's problem solving session.
  - Schedule 3: I will help you revise just like in schedule 2. But, we will try an IMO problem (related to today's lecture) during problem solving session.

Record the meeting…

Content so far…

L1:	Monovariants	I
L2:	Invariants	
L3:	Alternating-variants	
L4:	Inductive constructions	II
L5:	Greedy and RUST	
L6:	Counting in two ways	III
L7:	(Bonus) Polyhedron Formula	
L8:	(Bonus) Counting in graphs	
L9:	Injections and bijections	
L10:	Pigeonhole principle	IV
L11:	Continuity and descent	
L12:	Colouring and WLOG	
L13:	Combinatorial games	V
L14:	Combinatorial geometry	VI
L15:	Graph theory	VII
L16:	IMO Mock Test	

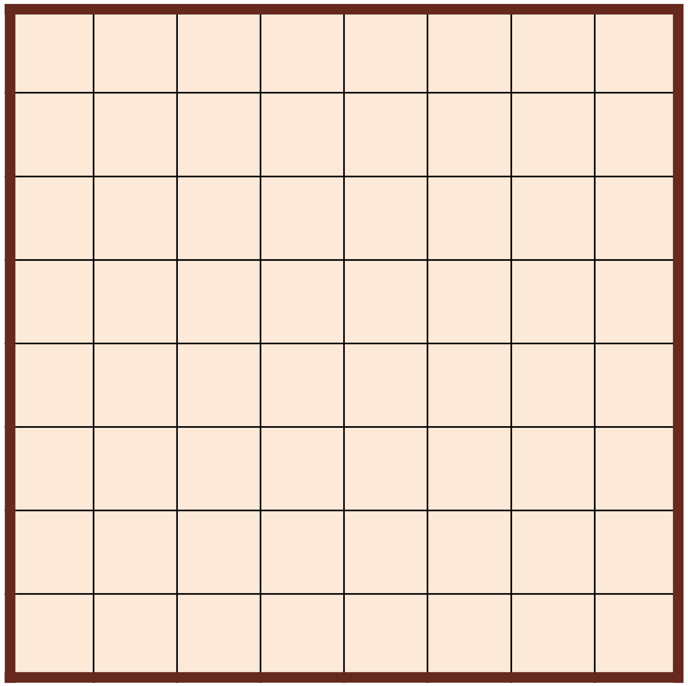
Lecture – 13

Combinatorial Games

Section – I

Placing Knights

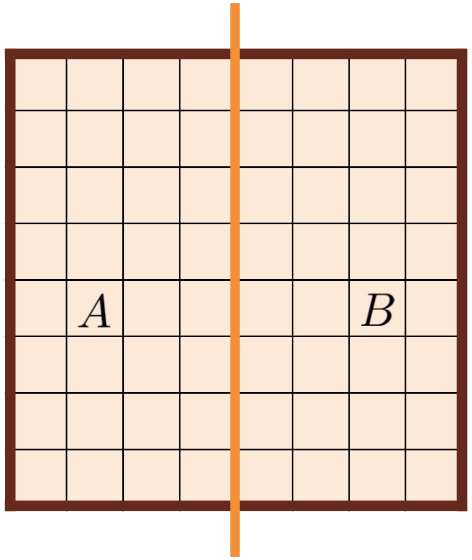
Alice and Bob are playing a game, alternatively placing chess-knights on an 8×8 chessboard starting with Alice. On Alice's turn, she places a white knight in an empty square that is not attacked by a black knight. On Bob's turn, he places a black knight in an empty square that is not attacked by a white knight. The first player who cannot place a knight loses. Who has the winning strategy?



Stay Alive

- Number of squares is even. So, if every square is filled, Alice loses.
- So, if Alice were to have a winning strategy, she must do something to make the game end earlier.
- But, for Bob, he just needs to stay alive.

Strategy for Bob. Say Alice put her knight on cell A in the previous turn. Let B be the reflection of A at the central vertical line. Bob simply places his knight on B.



Proof

- Suppose that Alice can place a knight on A, but Bob cannot place on B.
- Then, cell B is either occupied or attacked by a white knight placed in the previous turns (not the one at cell A).
- By induction on symmetry, A is either occupied or attacked by a black knight. Contradiction.

For Odd Chessboards

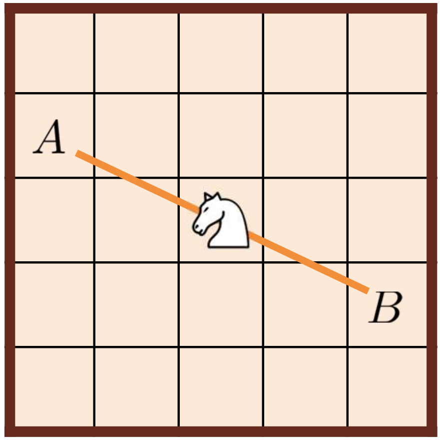
By reflection strategy, Bob can always win in $n \times n$ boards with n even.

Question: What if n is odd?

Intuition: In this case, the one who can win by staying alive is Alice. So, let's start by trying to make a copying strategy for Alice.

Strategy for Alice:

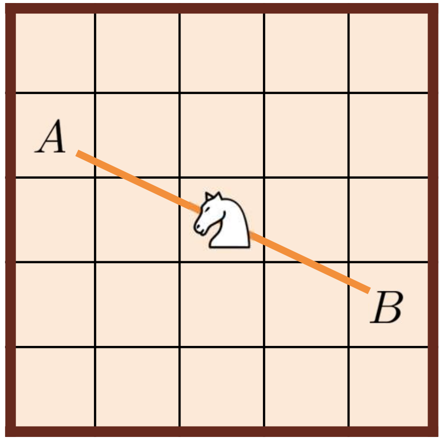
- Alice starts by putting her knight at the centre.
- For the rest, she just copies Bob by placing the knight at the reflection at the centre.



Proof that this strategy works is similar to previous.
(Actually, this strategy also works for Bob in the n even case)

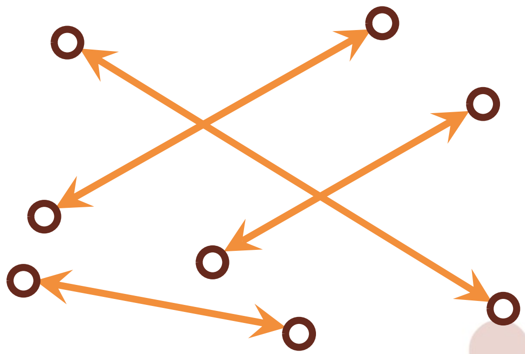
Winning by not Losing

The strategy used in previous problems are usually known as pairing or copying strategy.



One player plans out their strategy as follows:

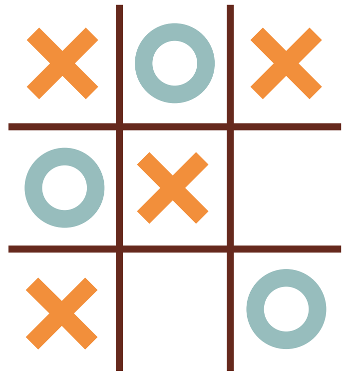
- Creates a pairing of the empty positions in some sense.
- Whenever the opponent plays in a position, play at the paired position.



Usually, the main goal of this strategy is to not lose. In games that eventually end, this serves as a winning strategy.

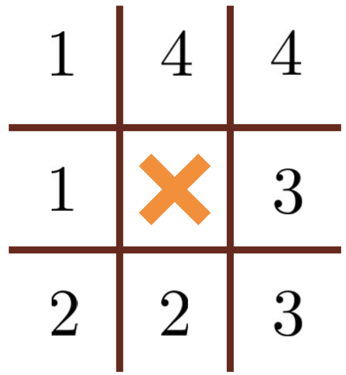
But, this strategy also has other applications.

How to not Lose in Tic-Tac-Toe



We all know Tic-Tac-Toe. Two players alternate to place X's and O's in a 3×3 grid. The first player to make three in a row/column/diagonal wins.

Well-known Fact: First player (say X) can play so that the second player (say O) never wins.



Strategy for X

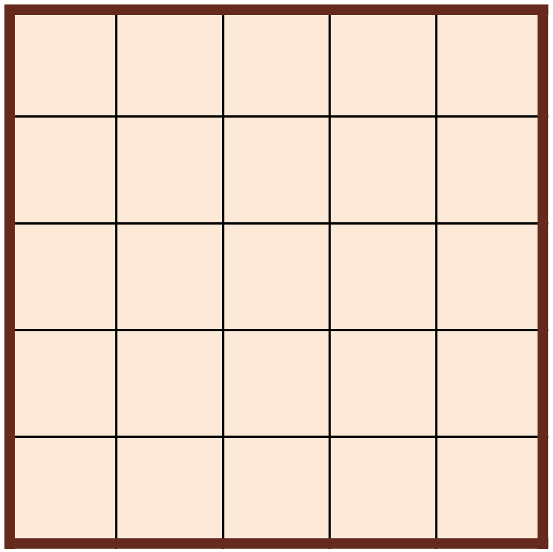
- Start at the centre.
- Pair up the cells as in the figure.
- Whenever O is filled in a cell, fill X in the paired cell.

Then, every row, column and diagonal has an X. So, O can never win.

Fact: Second player can also play so that they never lose. ← Exercise (not pairing)

5×5 Tic-Tac-Toe

Alice and Bob take turns to place X's and O's on an empty 5×5 board starting with Alice. On Alice's turn, she chooses an empty cell and places an X. On Bob's turn, he chooses an empty cell and places an O. Alice's goal is to get five X's in a row, column or diagonal. Bob's goal is to prevent this. Can Bob achieve his goal no matter how Alice plays?



5×5 Tic-Tac-Toe

Intuition:

- There are $5 + 5 + 2 = 12$ lines to block.
- There are 25 cells. So, we can create 12 pairs so that there is one O in each pair.

Question: Is it enough to cover all 12 lines?

Answer:

- Yes, just look at the picture to the right.
- If X is filled at the centre or if the paired cell is already filled with O, just ignore it and waste a move.

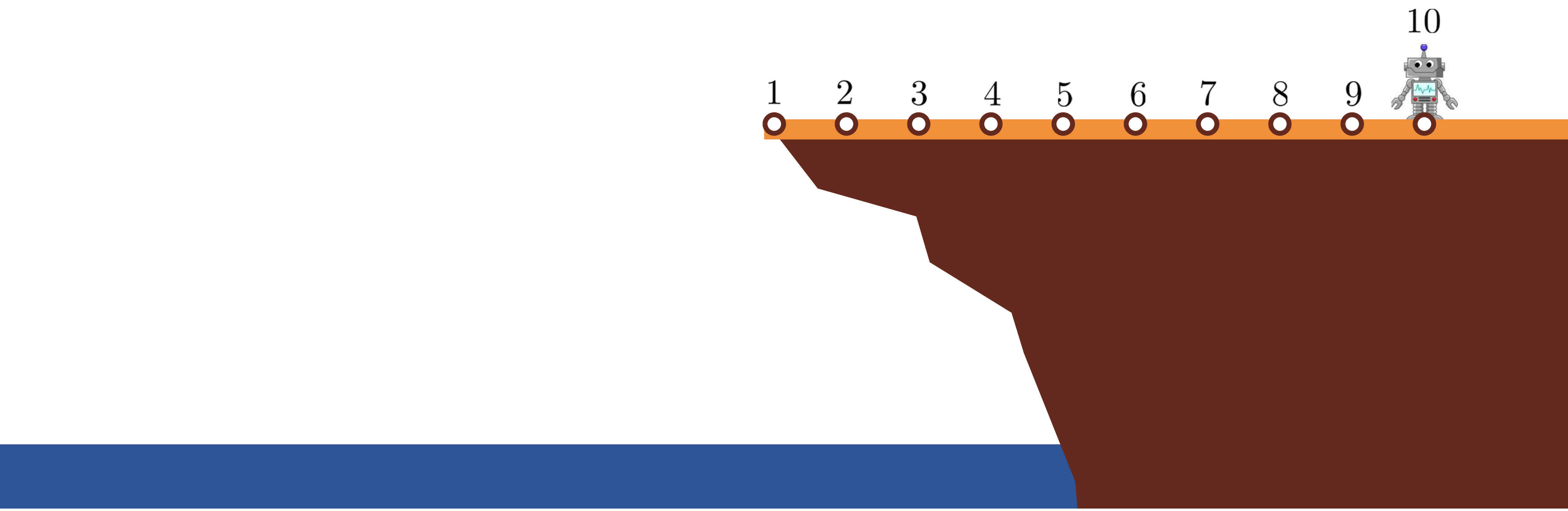
5	10	12	10	6
7	1	1	2	8
9	4		2	9
7	4	3	3	8
6	11	12	11	5

Note: Alice can also make sure that Bob doesn't get five-in-a-row using the exact same strategy.

Section – II

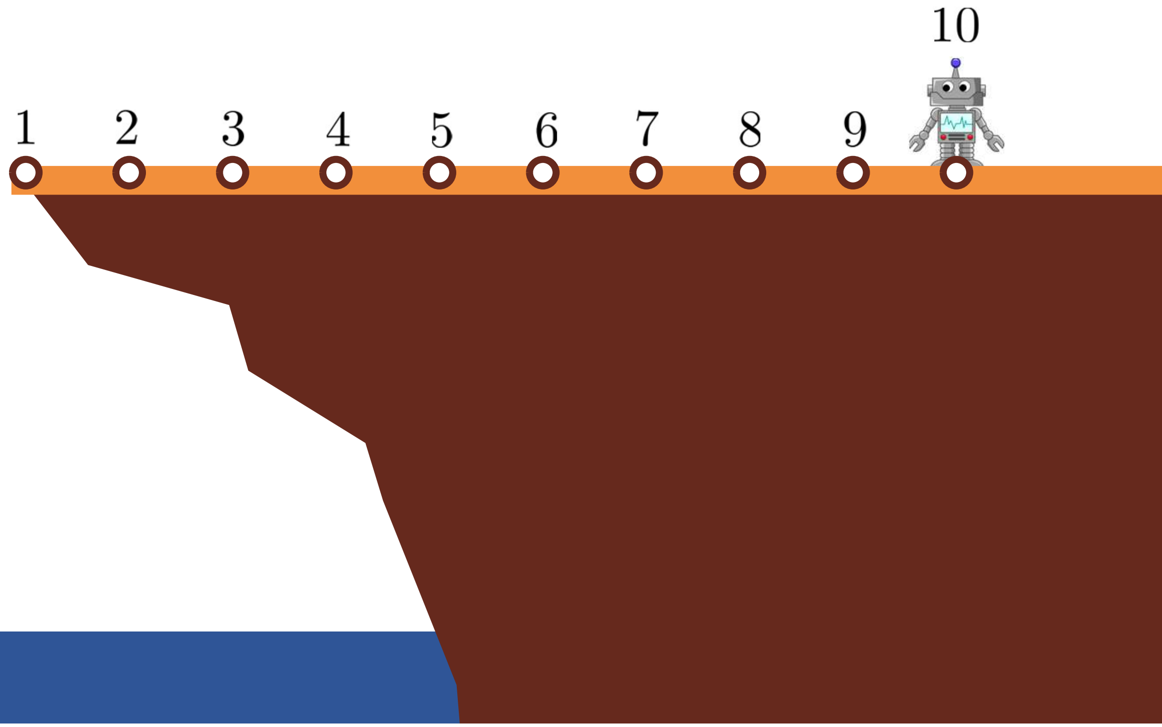
Subtraction Game

A robot is standing 10 steps away from falling down a cliff. Alice and Bob has a controller that can make a robot move 1, 2 or 3 steps towards the cliff. They take turns to use the controller, the player who makes the robot fall loses. If Alice starts, who has the winning strategy?



Work Backwards

- If the robot starts at 1, then Bob wins.
- So, if the robot starts at 2, 3, 4, then Alice wins.
- So, if the robot starts at 5, then Bob wins.
- So, if the robot starts at 6, 7, 8, then Alice wins.
- So, if the robot starts at 9, then Bob wins.
- So, if the robot starts at 10, then Alice wins.



The Death Traps

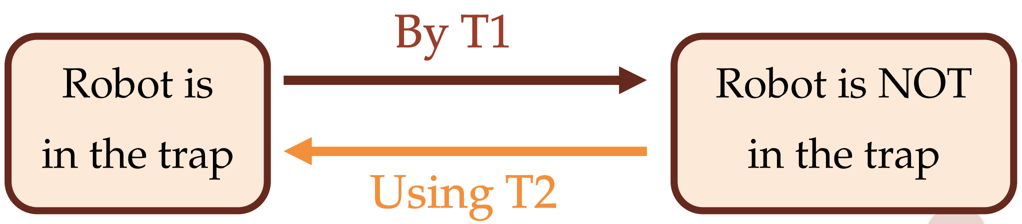
In previous solution, Bob wins if the robot starts on 1, 5, 9, 13, … .

These positions work as death traps.

Properties of Traps:

- Property T1: If the robot is in the trap, it will escape the trap in the next move.
- Property T2: If the robot is NOT in the trap, it can be put inside the trap inside the next move.

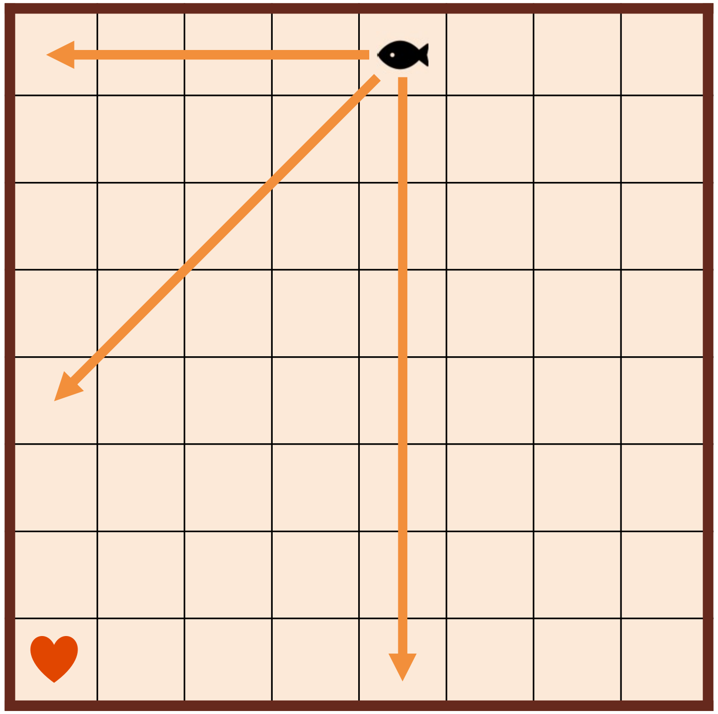
Strategy for "winning player" is to just keep putting the robot into the trap.



Then, robot stays in trap at the opponent's turn. Since the losing position (i.e. 1) is a trap, this means the opponent will lose.

Wythoff's Game

A fish starts somewhere on the 8×8 grid. Alice and Bob take turns to move the fish either to the west, south or southwest for any number of squares. There is a treat at the lower-left corner. The player who moves into the treat wins. For which starting squares does Alice have a winning strategy?



Find Traps Recursively

Remember the properties of trap-squares:

- Every move from trap-squares lead to non-trap squares.
- From non-trap squares, there is a move into a trap-square.

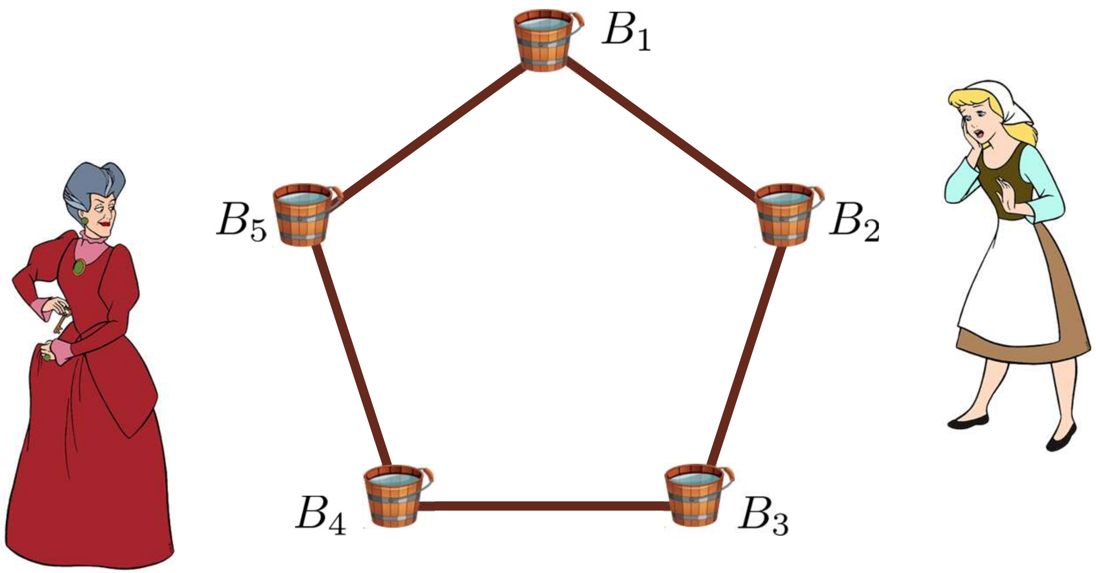
We want trap-squares to be losing positions (i.e. losing for the first player).

N	N	N	N	☠	N	N	N
N	N	N	N	N	N	N	N
N	N	N	☠	N	N	N	N
N	N	N	N	N	N	N	☠
N	N	N	N	N	☠	N	N
N	☠	N	N	N	N	N	N
N	N	☠	N	N	N	N	N
♥	N	N	N	N	N	N	N

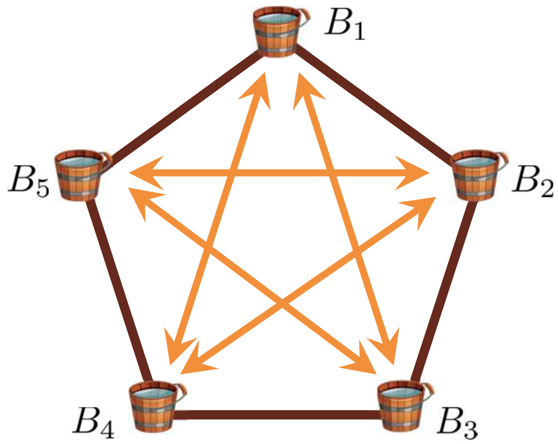
Section – III

Cinderella doing Chores (IMOSL 2009/C5)

Five identical empty buckets of 2-liter capacity stand at the vertices of a regular pentagon. Cinderella and her wicked stepmother go through a sequence of rounds: At the beginning of every round, the stepmother takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then, Cinderella chooses a pair of neighboring buckets, empties them into the river, and puts them back. Then, the next round begins. The stepmother's goal is to make one of these buckets overflow. Cinderella's goal is to prevent this. Can the wicked stepmother enforce a bucket overflow?



When can Cinderella Lose?



If Cinderella were to lose at round k, before she empties the buckets in that round, there must be two non-adjacent buckets with more than 1-litre of water in each.

So, Cinderella's goal is to achieve the following after her turn:

For any two non-adjacent buckets,
total water in them is at most 1-litre. ← Call such situation good

In symbols, this is what we want after her turn:

$$B_1 + B_3 \leq 1 \qquad B_2 + B_4 \leq 1 \qquad B_3 + B_5 \leq 1 \qquad B_4 + B_1 \leq 1 \qquad B_5 + B_2 \leq 1$$

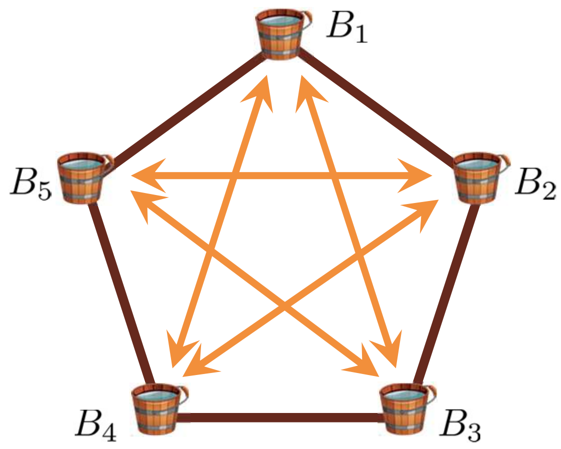
Since she just finished her turn, we can WLOG, assume that $B_4 = B_5 = 0$

Preserving the Good State

Before Stepmother's turn: $B_1 + B_3 \leq 1$ $B_2 \leq 1$ $\cancel{B_3 \leq 1}$ $\cancel{B_1 \leq 1}$ $\cancel{B_2 \leq 1}$

Dangerous

After Stepmother's turn: $B_1 + B_3 + B_4 + B_5 \leq 2$ $B_2 \leq 2$



Cinderella's Strategy:

- Immediately empty B_2.
- Now, we don't need to worry about (B_2, B_5) and (B_2, B_4).
- Note that one of the followings is true:

$$B_1 + B_4 \leq 1 \qquad B_3 + B_5 \leq 1$$

- In first case, empty B_3. In second case, empty B_1.
- Then, Cinderella can preserve the good situation.

Summary

- Section I: Pairing and mirroring strategies
  - These focus on winning by not losing (e.g. the knights game).
  - Also works as an easy strategy to guarantee something (e.g. there is always one X in these two cells).

- Section II: Positional analysis (only for symmetric games i.e. both players have same rule-set)
  - By working backwards, we can recursively figure out the trap-positions.
  - The go-to strategy is to just keep pushing the opponent into traps.

- Section III: Maker-breaker Games
  - Finding the invariant (good position) that a player can preserve is usually a go-to strategy in these kind of games.

That’s it for this lecture.

See you tomorrow!