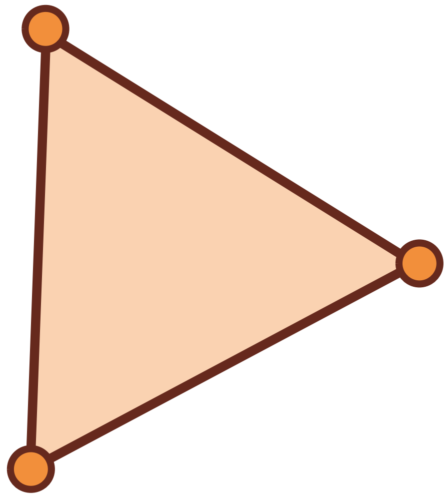
We will begin at 8:35 MMT

Read this problem while we wait…

Show that there is no equilateral triangle all of whose vertices are points with integer coordinates.

What if we replace 'equilateral triangle' above with a regular n-gon?



Record the meeting…

Some Housekeeping

Problem set 5 is due tomorrow! Pigeonhole principle is quite important, so please submit.

We will have Mock IMO on 19 July (Monday) and 21 July (Wednesday). This is mandatory for IMO team members. Exact date, time and details are in announcements.

Mock IMO Format

- Each day contain four problems and lasts 4.5 hours. First two problems are easier than the last two problems. All four categories: A, C, G, N will be present inside each colour.

- For combinatorics, easier means at most 3-diamonds (much easier than IMO1/4). Harder means at least 4-diamonds (can be up to IMO1/4).

Day 1	Day 2
Problem 1. Easier	**Problem 5.** Easier
Problem 2. Easier	**Problem 6.** Easier
Problem 3. Harder	**Problem 7.** Harder
Problem 4. Harder	**Problem 8.** Harder

Content so far…

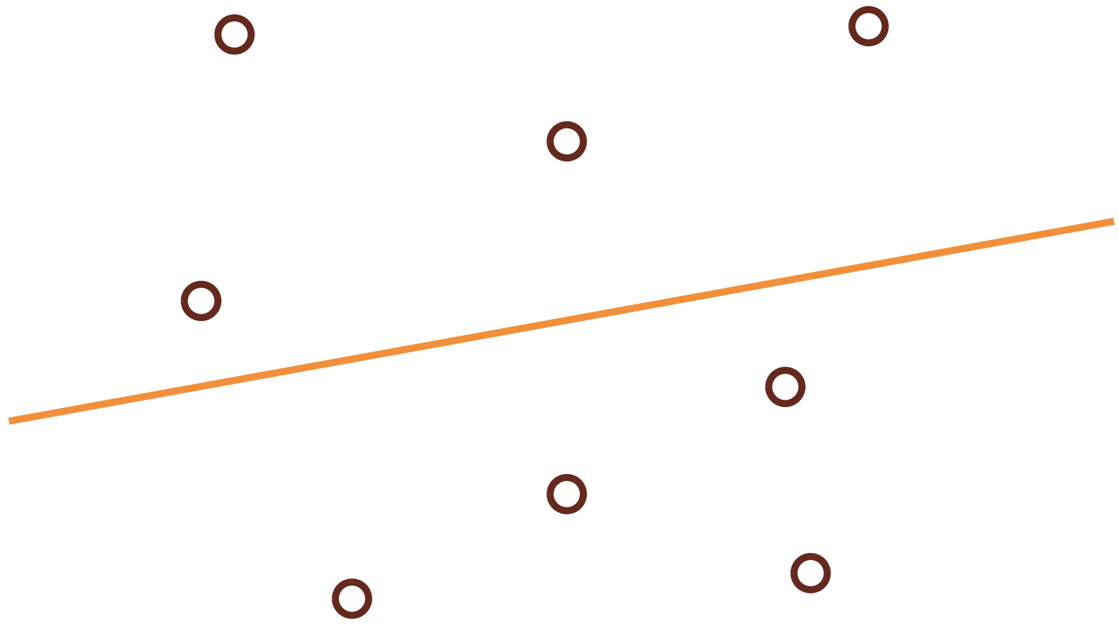
L1: Monovariants L2: Invariants L3: Alternating-variants	I
L4: Inductive constructions L5: Greedy and RUST	II
L6: Counting in two ways L7: (Bonus) Polyhedron Formula L8: (Bonus) Counting in graphs L9: Injections and bijections	III
L10: Pigeonhole principle L11: Continuity and descent L12: Leveraging symmetry	IV
L13: Combinatorial games	V
L14: Combinatorial geometry	VI
L15: Graph theory L16: IMO Mock Test	VII

Lecture – 11

Discrete Continuity and Descent

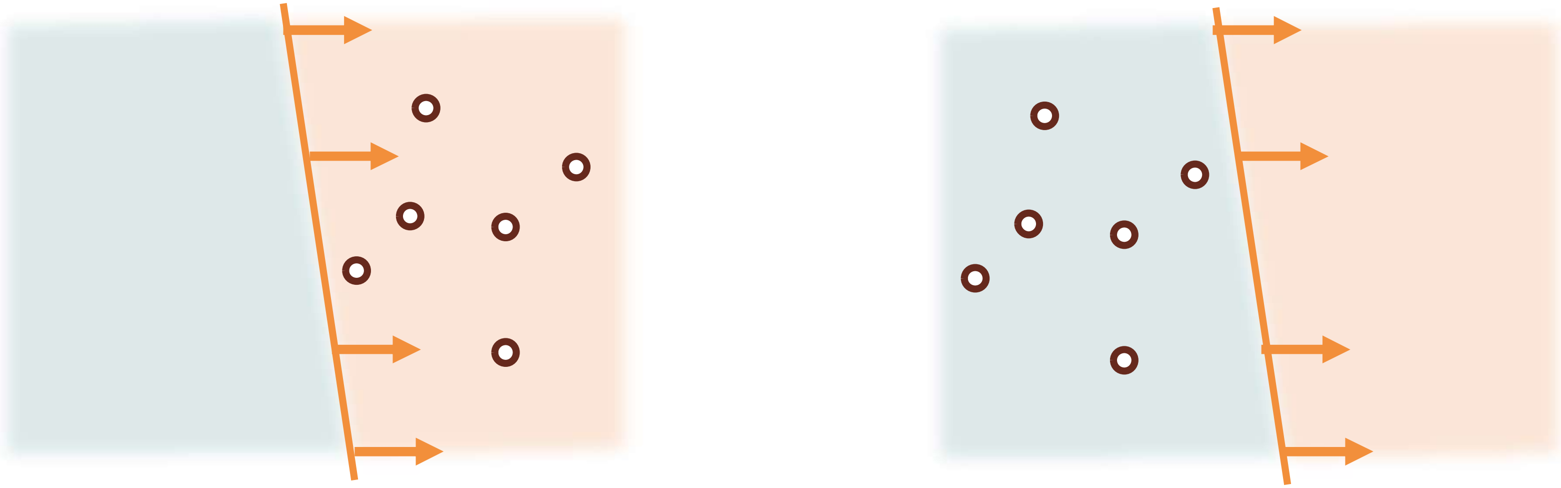
Halving Line

Given $2n$ points in the plane, can we always draw a line so that exactly half of the points lie strictly on one side of the line?



Halving Line

Idea: Draw a random line so that all the points lie on one side. Then, slowly slide the line.



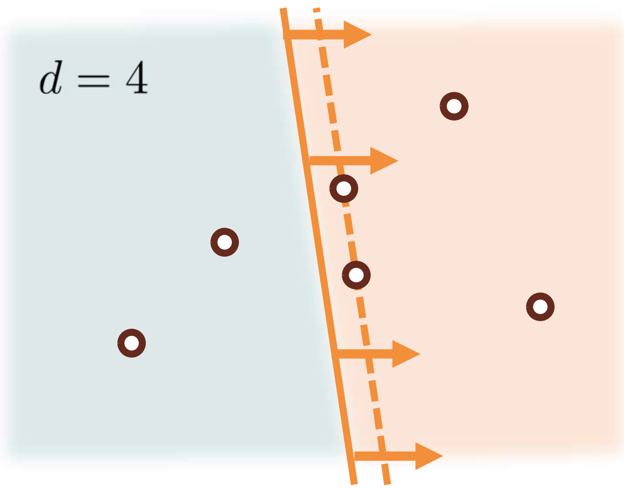
Colour the “left” side of the line teal, and the “right” side orange.

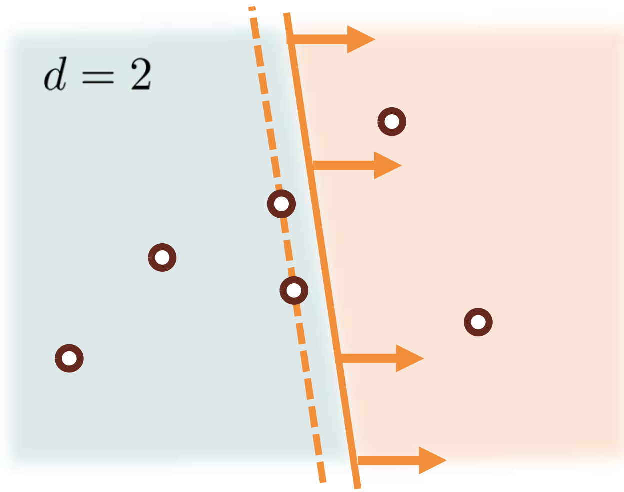
Let d be the number of points in orange region.

Then, we started with $d = 2n$ and ended with $d = 0$. We want show that $d = n$ somewhere.

Bad Cases

But, this might not happen, value of d can "skip" n while slowly decreasing to 0.





This bad case can happen exactly when our line is parallel to one of the $\leq \binom{2n}{2}$ segments determined by the points. This allows our line to "skip" two points at a time.

How do you fix this? ← Easy, choose our line so that it is not parallel to any of the segments.

The Full Solution

- Choose a direction m not parallel to any of the segments determined by the $2n$ points (such direction exists).

- Let l be a line in the direction of m so that all of the $2n$ points lie on the "right" side of l.

- Translate the line to the "right" until all the $2n$ points get to the "left" side.

- Let d be the number of points on the "right" side of l. We started with $d = 2n$ and ended up with $d = 0$.

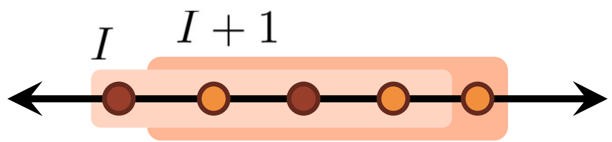
- Since l is not parallel to any of the segments, d changes by at most one each time. So, $d = n$ somewhere along the way.

Brown and Orange Integers (Italy MO 2013/P3)

Each integer is coloured with one of the two colours: brown or orange. It is known that, for every finite set A of consecutive integers, the absolute value of the difference between the number of brown and orange integers in the set A is at most 1000. Prove that there exists a set of 2000 consecutive integers in which there are exactly 1000 brown and 1000 orange numbers.

Discrete Continuity?

Observation: If we take an interval I of 2000 consecutive integers, the number of brown integers changes by at most one if we translate I by one.



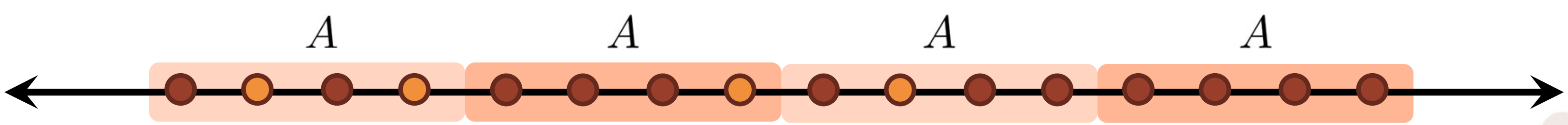
What does this give us?

To do: Find two intervals of 2000 consecutive integers:

- Type A: one with at least 1000 brown points,
- Type B: one with at most 1000 brown points.

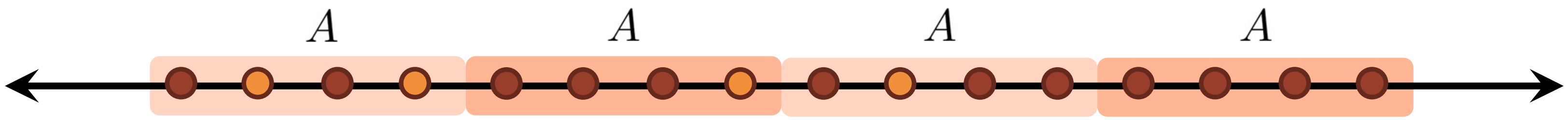
Estimation-claims are easier to prove than exact-claims.

Suppose the otherwise, then only one type is present.



Resolving the Two Types

- Say every interval of 2000 integers is of type A (similar proof holds for type B).



- Now, divide $\mathbb{Z}$ into many disjoint intervals of 2000 integers.
- If one of those has exactly 1000 brown integers, we are done.
- So, suppose that all of them have less than 1000 brown integers.
- Then, take 1001 consecutive such intervals. That collection of 1001×2000 integers contain at least 1001 more browns than oranges, contradiction.

(for self-study) Another way to find two types

To do: Find two intervals of 2000 consecutive integers:

- Type A: one with at least 1000 brown points,
- Type B: one with at most 1000 brown points.

This looks a lot like pigeonhole principle.

Pigeons are points, pigeonholes are intervals of 2000 consecutive integers.

Consider a big interval I of $2000n$ consecutive integers (n is any positive integer), divide it into n disjoint intervals with 2000 integers each.

Number of brown points in I is between $1000n - 500$ and $1000n + 500$.

So, average number of brown points in each disjoint interval between $1000 - \dfrac{500}{n}$ and $1000 + \dfrac{500}{n}$

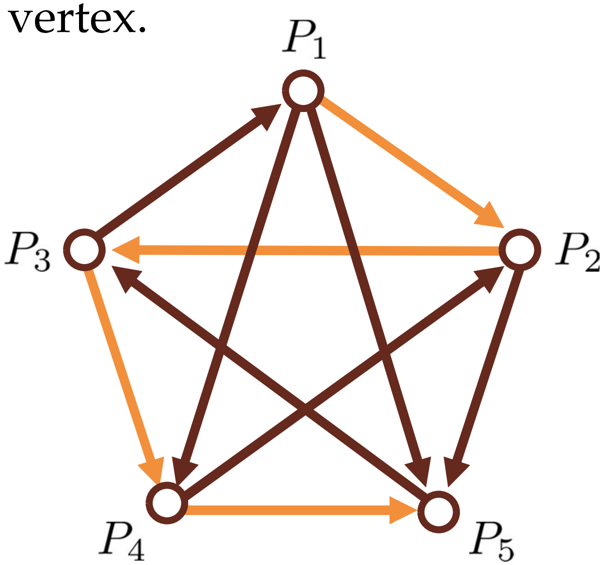
Hence, if $n > 500$, one such interval contains at least 1000 brown and one such interval contains at most 1000 brown.

How not to rank a tournament

Suppose that n people participate in a round-robin tennis tournament. Each match is played by two people, and there are no draws. Prove that, after all the matches have been played, it is possible to arrange the players in a line: $P_1, P_2, P_3, \ldots, P_n$ so that P_k beats P_{k+1} for all positive integers $1 \leq k < n$.

These digraphs are called tournaments.

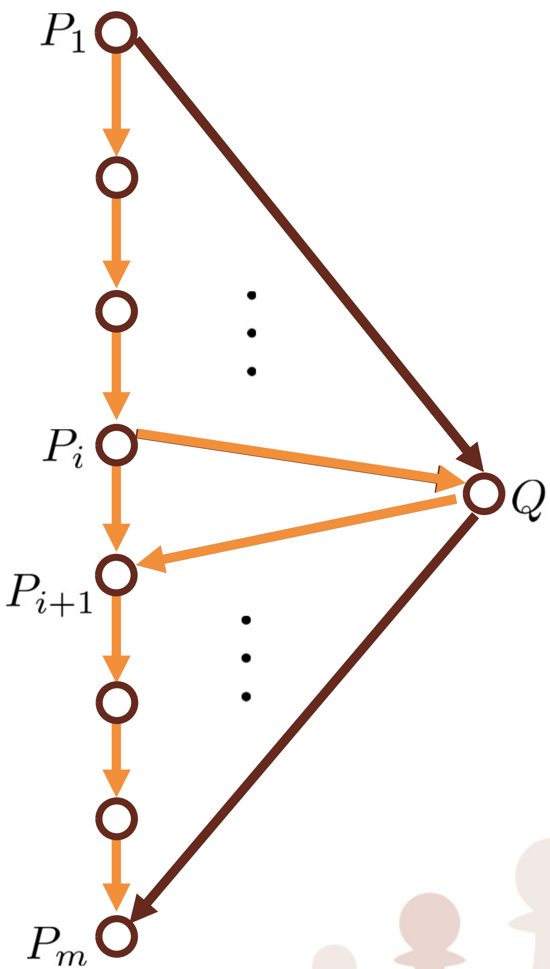
Graph Formulation: Show that in a directed graph with underlying graph being K_n, there exists a directed path which contains every vertex.



How not to rank a tournament

Idea: Take the longest directed path. Show that it contains every vertex.

- Let $P_1 \to P_2 \to \cdots \to P_m$ be the longest directed path. Suppose to the contrary that $m < n$.
- Let Q be the vertex outside the path.
- Then, $P_1 \to Q$ and $P_m \leftarrow Q$.
- So, there are two consecutive persons P_i and P_{i+1} such that $P_i \to Q$ and $P_{i+1} \leftarrow Q$.
- Now, $P_1 \to \cdots \to P_i \to Q \to P_{i+1} \to \cdots \to P_m$ is a longer path, contradiction.



Irrationality of $\sqrt{2}$

I am sure that many of you have seen the proof for the irrationality of $\sqrt{2}$.

- Suppose that $\sqrt{2} = p/q$ for some positive integers p and q.
- Then, $p^2 = 2q^2$ and so p is even, let $p = 2p_1$ for some $p_1 \in \mathbb{Z}_{>0}$.
- Substituting, we get $2p_1^2 = q^2$. So, q is even, let $q = 2q_1$ for some $q_1 \in \mathbb{Z}_{>0}$.
- Now, we have $\sqrt{2} = p_1/q_1$. So, we can repeat the argument to construct

$$(p_2, q_2), \quad (p_3, q_3), \quad (p_4, q_4), \quad \cdots$$

- But, $p > p_1 > p_2 > \cdots$ and they are positive integers!

Hence, we get a contradiction.

$$(p, q) \downarrow (p_1, q_1) \downarrow (p_2, q_2) \downarrow (p_3, q_3) \downarrow$$

Gets smaller in some sense…

Irrationality of $\sqrt{2}$

This maybe a little bit different from what you have seen. Normally, the proofs for irrationality of $\sqrt{2}$ go like this:

- Suppose that $\sqrt{2} = p/q$ for some relatively prime positive integers p and q.

  $$\ldots \text{ same argument as before} \ldots$$

  Therefore, p and q are both even and hence not relatively prime. We get a contradiction.

- Well, we can also do something like this: Let $\sqrt{2} = p/q$ for some positive integers p and q so that $p+q$ is as small as possible.

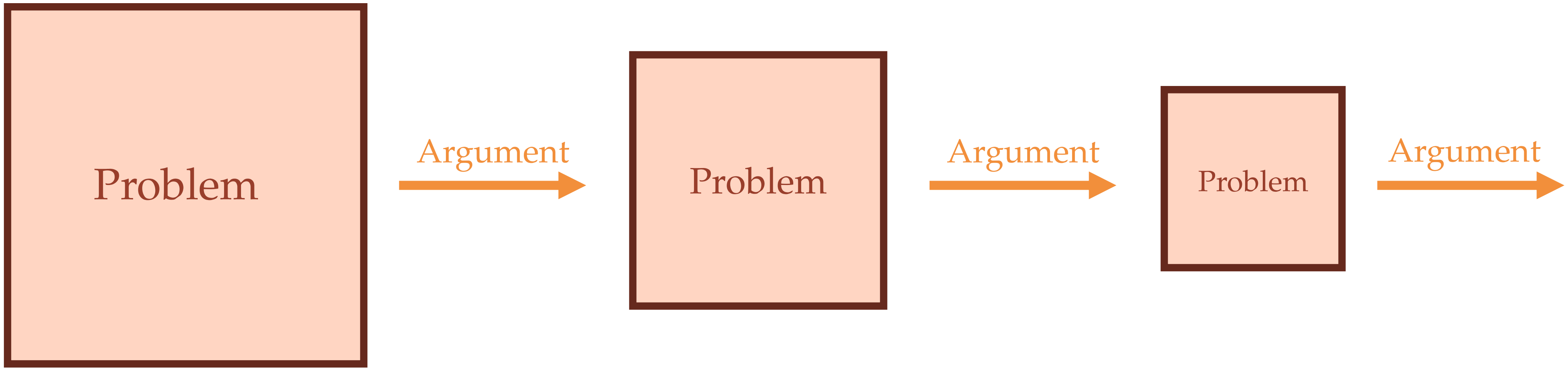
  $$\ldots \text{ same argument as before} \ldots$$

  Therefore, $\sqrt{2} = p_1/q_1$ yet $p_1 + q_1$ is smaller than $p+q$, contradiction.

The only important thing is "by repeating some argument, you can keep reducing some quantity."

The Infinite Descent

The idea is basically identical to process-termination by monovariants, and the argument resembles induction-style (except it doesn't stop this time).



Partitions Again (India TST 2003)

Let n be a positive integer. The set $\{1, 2, 3, \ldots, 3n\}$ is partitioned into subsets A, B and C such that $|A| = |B| = |C| = n$. Prove that there exist $x \in A$, $y \in B$ and $z \in C$ such that one of x, y and z is the sum of the other two.

Examples

- If $n = 1$, the sets A, B and C are $\{1\}$, $\{2\}$ or $\{3\}$. So, the conclusion is obvious.

- For $n = 3$, suppose

$$A = \{1, 2, 6\}, \qquad B = \{3, 7, 9\}, \qquad C = \{4, 5, 8\}.$$

Then, we can see that $1 \in A$, $9 \in B$ and $8 \in C$ satisfy the desired condition.

Prologue to the Solution

Suppose the conclusion is false: difference between two elements from different sets A, B or C does not lie inside the other set.

Suppose WLOG that $1 \in A$ and $2, 3, \ldots, k-1 \in A$ for some positive integer $k \geq 2$.

Suppose WLOG that $k \in B$. Then, we have the following properties:

- Property X: Elements of A and C cannot differ by k.
- Property Y: Elements of B and C cannot differ by $1, 2, 3, \ldots, k-1$.

Let $m \in C$. Now, let's start the property chasing!

For example: $m-1$ cannot be in B. $m-k$ cannot be in A.

Chase Chase Chase Chase

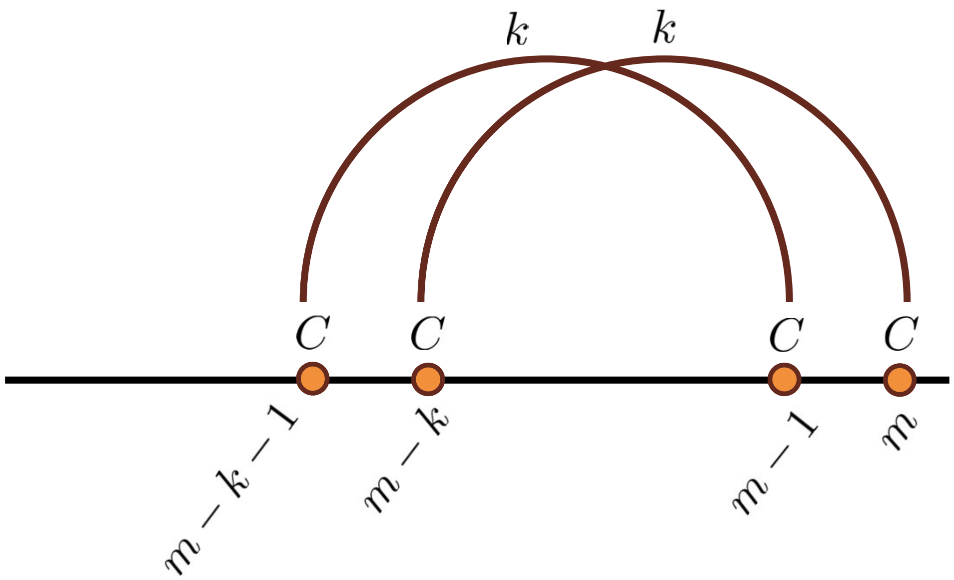
- Property X: Elements of A and C cannot differ by k.
- Property Y: Elements of B and C cannot differ by $1, 2, 3, \ldots, k-1$.

Let $m \in C$. Then, $m-1$ cannot be in B. So, $m-1 \in A$ or $m-1 \in C$.

Consider the case: $m-1 \in C$

- By property X, $m-k$ cannot be in A.
- By property Y, $m-k$ cannot be in B.
- So, $m-k \in C$.

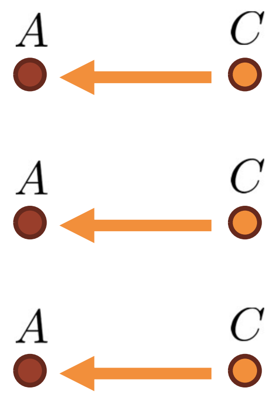
- By property X, $m-k-1$ cannot be in A.
- By property Y, $m-k-1$ cannot be in B.
- So, $m-k-1 \in C$.



Can you see an infinite descent?

Chase Chase Chase Chase

- So, if $m \in C$, then we must have $m - 1 \in A$.
- Thus, every element of C was succeeded by an element of A.
- Since A and C have same size, every element of A is also succeeded by an element of C.
- Since $k - 1 \in A$, this implies that $k \in C$.
- But, $k \in B$. Contradiction!



Today’s Techs

- Discrete continuity: move one at a time, something changes at most one at a time.
- Discrete continuity (ver.2): if a sequence of objects contain two types of objects, and if the first and last object have different types, then there is a place where the two different types are adjacent.
- Infinite descent: You get a smaller version of the problem/case you are considering. You can apply the same argument to that version.
- Assume to the contrary: If what we want to prove is an existence statement, assuming the contrary gives us a property along the lines of “whenever … we have …”.
- Property chasing: When you have a lot of statements of the form “whenever … we have …”, you can study the position by chasing the properties and separating cases if necessary.

That's it for this lecture.

See you tomorrow!