We will begin at 1:05 PM MMT

Read this problem while we wait…

Put a finite number of random people into a room. Prove that two of them is friends with the same amount of people in the room.

(Friendship is mutual i.e. if A is friends with B, B is also friends with A).

Record the meeting…

Some Housekeeping

- For convenience, I decided to extend the deadline of (optional) Pset4 for two more weeks.

Content so far…

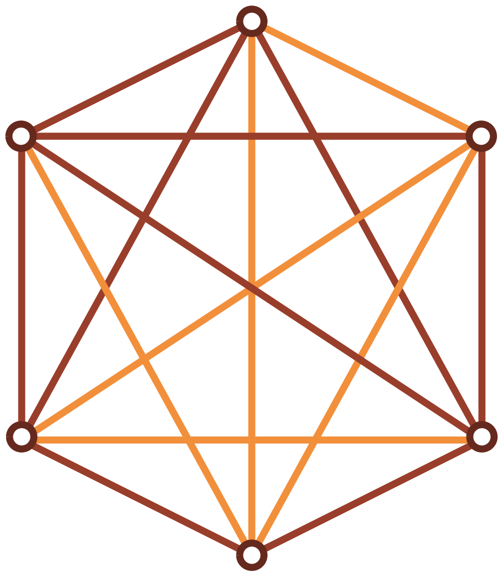
Lecture – 10

Pigeonhole Principle

The Six-person Folklore

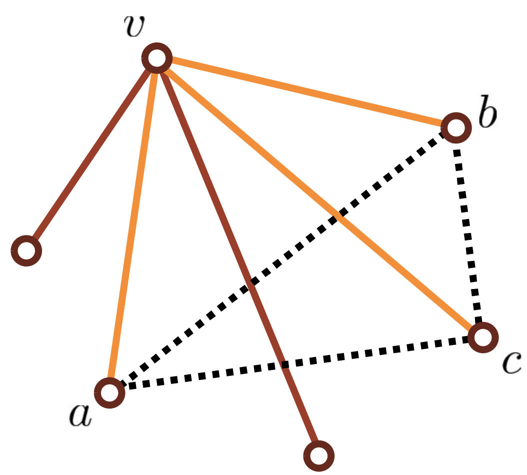
Prove that whenever there are six people in the room, there exist three people that know each other, or three people that don't know each other.

Graph Formulation: If you colour the edges of K_6 in two colours (brown/orange), then there is either a brown triangle or an orange triangle.



The Six-person Folklore

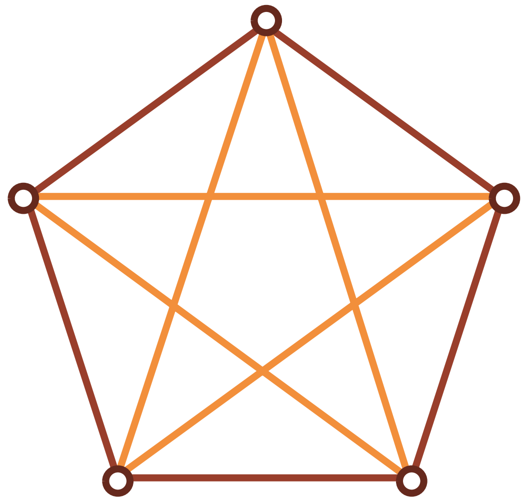
- Look at a specific vertex v. There are 5 edges incident to v.
- So, at least three of these edges say va, vb and vc must have the same colour (say orange).
- If any one of ab, bc or ca is orange, then we will have an orange triangle. So, we are done in this case.
- Otherwise, abc is a brown triangle, so we are also done.



The Six-person Folklore

Six is the smallest number for which we can achieve this result.

The following shows a colouring of K_5 with no monochromatic triangle.



The Pigeonhole Principle

Theorem. Suppose you have $n+1$ pigeons and there are n pigeonholes. If all the pigeons fly into these pigeonholes, then there is a hole containing at least 2 pigeons.

Theorem (Stronger). Suppose you have M pigeons and there are n pigeonholes. If all the pigeons fly into these pigeonholes, then there is a hole containing at least M/n pigeons.

Proof. Otherwise, every hole contains less than M/n pigeons. So, there will be less than $n \cdot M/n$ pigeons to begin with, contradiction.

Examples.

- If I distribute 100 pens to 8 students, at least one student will get 12 pens.
- If I place 9 rooks onto an 8×8 chessboard, then there are two attacking rooks.
- Within any five integers, you can select two of them whose difference is divisible by 4.

The Pigeonhole Principle

Theorem (Even stronger). Suppose there are real numbers $a_1, a_2, \ldots, a_n$ with average μ. Then, one of these numbers is at least μ and (possibly another) one of these is at most μ.

Proof. Let $m = a_i$ and $M = a_j$ be minimum and maximum in $a_1, a_2, \ldots, a_n$. Note that

$$m = \frac{m + \cdots + m}{n} \leq \frac{a_1 + \cdots + a_n}{n} \leq \frac{M + \cdots + M}{n} = M.$$

Therefore, $a_i \leq \mu \leq a_j$.

So, Pigeonhole principle is basically saying that average is between maximum and minimum.

Example: If a graph on 10 vertices has 26 edges, one of the vertices has degree ≥ 6 and one of the vertices has degree ≤ 5.

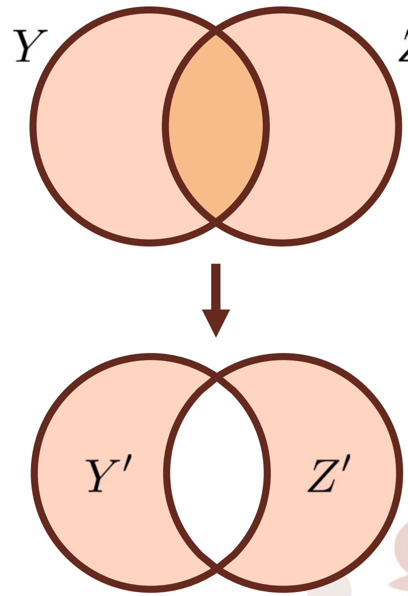
Because sum of degrees is 52, so the average degree is 5.2.

Same Sum Subsets (IMO 1972/P1)

Let X be a subset of $\{1, 2, 3, \ldots, 99\}$ of size 10. Prove that there are disjoint non-empty subsets Y and Z of X such that sum of the elements in Y is the same with that of Z.

Example. If $X = \{2, 3, 10, 15, 20, 22, 38, 40, 50, 76\}$, then we may select $Y = \{2, 3, 15\}$, $Z = \{10, 20\}$.

Question: Do we really need Y and Z to be disjoint? That is, what if we have Y and Z with same sum, but intersecting, what can we do?



Answer: Easy, just remove the intersection from both Y and Z. The resulting parts will still have the same sum, but now disjoint.

So, we can just ignore the disjoint-condition.

How to find Y and Z

Pigeons should be subsets of X. Pigeonholes should be sum of the numbers.

Pigeons: How many non-empty subsets of X are there? ← $2^{10} - 1 = 1023$

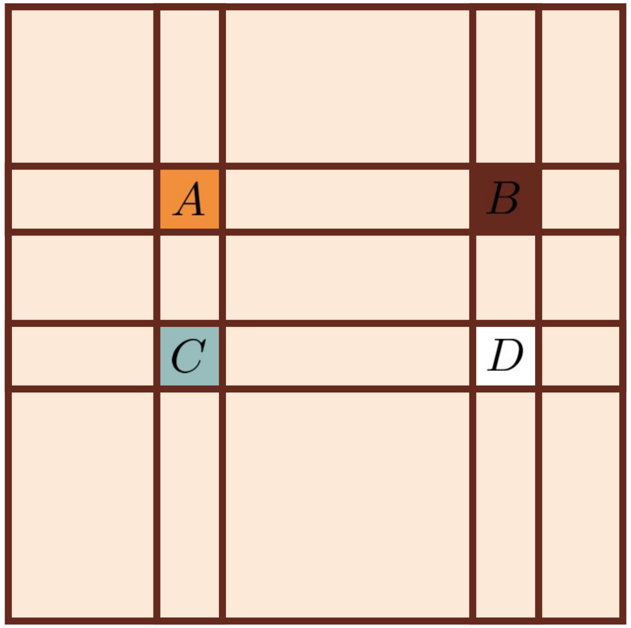
Holes: How many possible sums are there? ← 975

- Minimum possible sum is 1.
- Maximum possible sum is $99 + 98 + 97 + \cdots + 90 = 975$.

So, two of the non-empty subsets of X must have the same sum! Just take them to be Y and Z, then remove the intersection.

Colourful Rectangle (All Russian MO 2000)

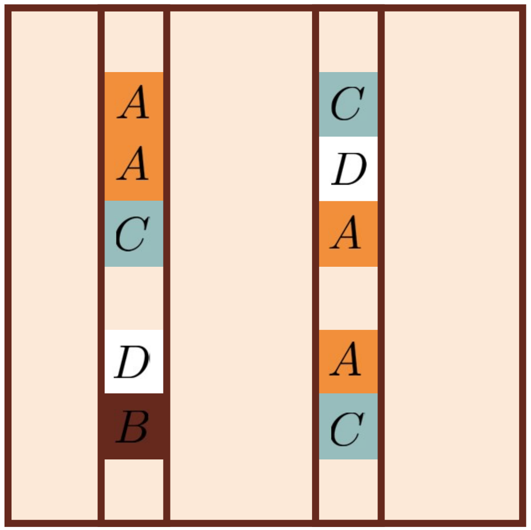
An 100×100 grid is coloured with four colours A, B, C and D so that there are 25 cells of each colour in each row and each column. Prove that there are two rows and two columns such that the four cells in their intersection have different colours.



Solution Sketch

Our approach will follow these two steps:

- **Step 1:** Find 2 columns that intersects a lot of rows at different colours.
- **Step 2:** Select two of those rows so that all four intersections are differently coloured.

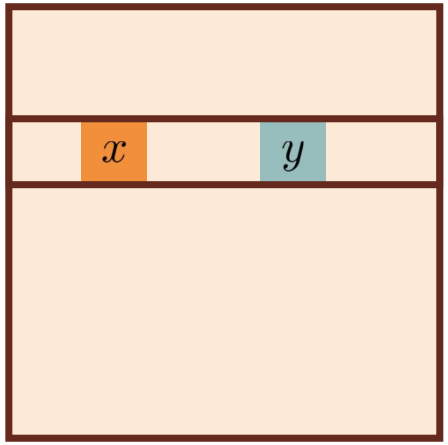


Step 1: Finding Two Columns

Pigeonholes: Pairs of columns $\binom{100}{2} = 50 \cdot 99$

Pigeons: Pairs of cells $\{x, y\}$ such that x and y lie in same row and are coloured differently (call them colourful pairs).

$$100 \cdot \binom{4}{2} \cdot 25 \cdot 25$$



Average number of pigeons in each hole is

$$\frac{100 \cdot 6 \cdot 25 \cdot 25}{50 \cdot 99} = \frac{100}{99} \cdot 75 \ > 75.$$

Thus, there is a pair of columns containing 76 colourful pairs.

Step 2: Choosing 2 out of 76

- Suppose to the otherwise. Then, any two of these 76 colourful pairs share a colour.

- Let's say that one of these pairs is coloured with $\boxed{A \text{ and } B.}$

- Colour A cannot appear more than 50 times in the columns. So, there is a colourful pair that doesn't use colour A. Say this pair is coloured with $\boxed{C \text{ and } B.}$

- Similarly, there is a pair that doesn't use colour B. This pair is coloured with either A and D or with A and C. We are done if it is A and D. So, it is coloured with $\boxed{A \text{ and } C.}$

- Then, all pairs use colours A, B and C only. ← Up to this step is just property-chasing (i.e. straightforward, but non-trivial).

- Since there are ≥ 152 cells in our pairs, one of the colours is used more than 50 times. That colour is used more than 25 times in a column. Contradiction.

Infinite Pigeons

Theorem. Suppose there are infinitely many pigeons and a finite amount of pigeonholes. If all the pigeons fly into these pigeonholes, then there is a hole containing infinitely many pigeons.

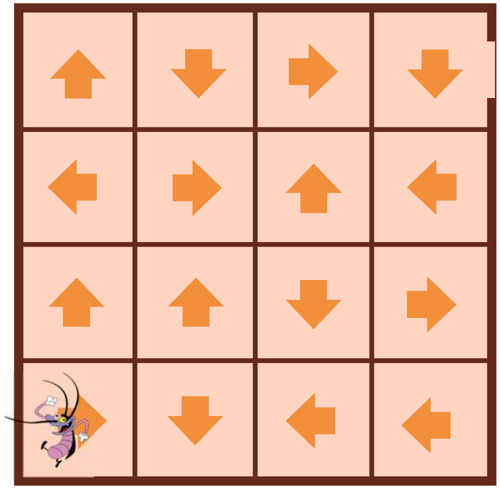
Example

Decimal representation of a rational number is either finite or eventually terminates.

Proof: Problem set 5.

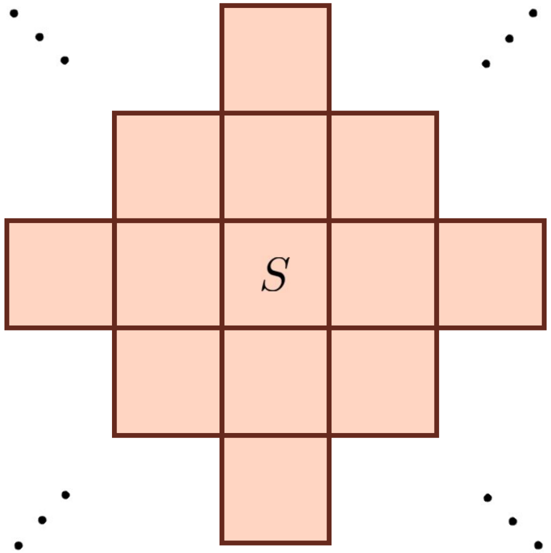
A Bug in a Maze

A 100×100 board is divided into unit squares. In every square there is an arrow that points up, down, left or right. The board square is surrounded by a wall, except for the right side of the top right corner square. An insect is placed in one of the squares. Each second, the insect moves one unit in the direction of the arrow in its square. When the insect moves, the arrow of the square it was in moves 90 degrees clockwise. If the indicated movement cannot be done, the insect does not move that second, but the arrow in its squares does rotate. Is it possible that the insect never leaves the board?



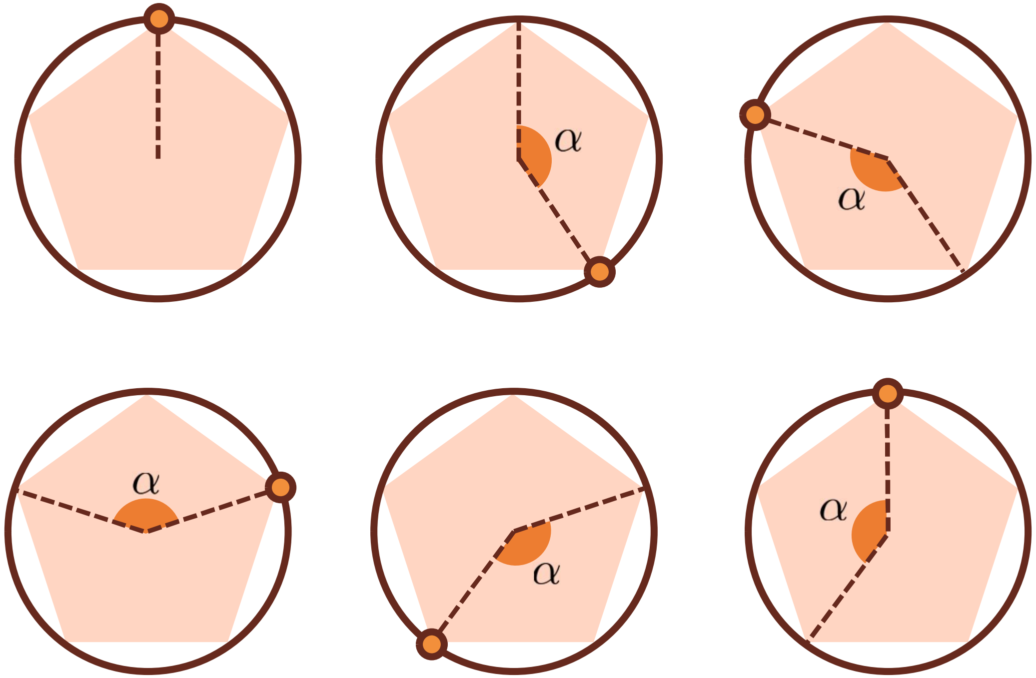
A Bug in a Maze

- Suppose the insect never leaves.
- Then, one of the squares, say S will be visited infinitely many times.
- So, neighbours of S will be visited infinitely many times.
- $\dots$
- So, the top-right corner will be visited infinitely many times.
- But, he will escape in at most four visits to the top-right corner, contradiction.



Rational Rotations

Let $\alpha > 0$ be a rational number. A point P first sits at some point on a circle. Every second, P is rotated by α degrees clockwise. What is the set of possible positions of P look like?



In this picture, $\alpha = 144°$.

Since $5\alpha = 360° \times 2$, point P returns to the start at the 5^{th} move.

In fact, $\alpha, 2\alpha, 3\alpha, 4\alpha, 5\alpha$ form the five multiples of $72°$ mod $360°$.

Thus, P traces along the vertices of a regular pentagon.

Rational Rotations

In general, let $\alpha = p/q$ where p and q are positive integers.

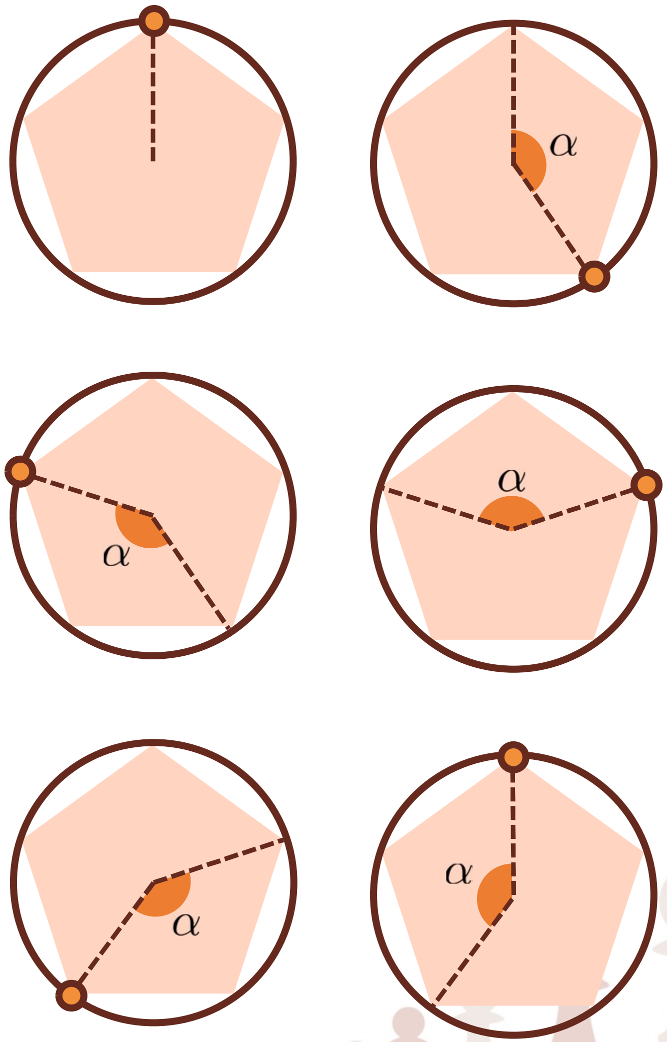
Consider the sequence: α, 2α, 3α, ... Is there an integer multiple of 360 in this sequence?

Answer: Yes, for example, $360q\alpha$. ← So, position of P is periodic.

What is the first time that P returns to the original position?

Answer: Exercise (this is more of a number theory question).

But, it is true that P traces the vertices of some regular polygon.

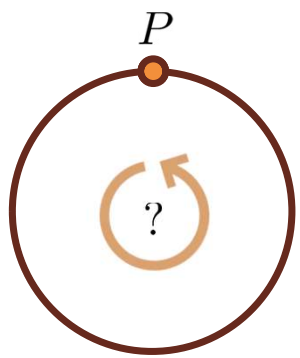


Irrational Rotations

Let $\alpha > 0$ be an irrational number. A point P first sits at some point on a circle. Every second, P is rotated by α degrees clockwise. What is the set of possible positions of P look like?

Question: Is the position of P periodic like in rational case?

Answer: No.



Proof: If it is periodic, then $N\alpha$ will be divisible by 360 for some positive integer N.

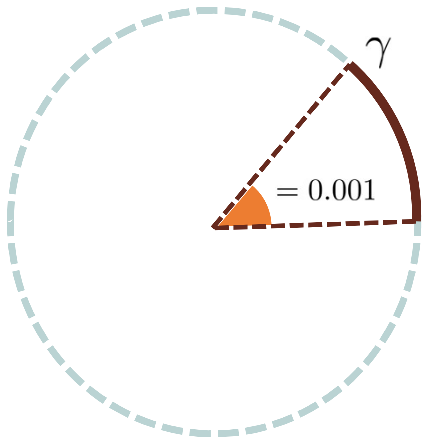
This means that $N\alpha = M$ for some integer M.

This contradicts the fact that α is irrational.

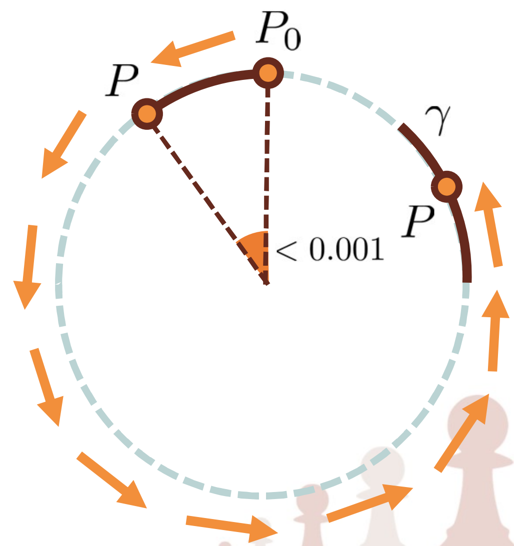
Irrational Rotations

Thus, positions of P are infinite.

In fact, given any arc γ of the circle, no matter how small it is, we can guarantee that P will eventually lie inside the arc γ.



Proof: Let P_0 be the starting position of P. For illustration purposes, let the degree measure of γ be 0.001 (general case is similar). Our proof will go in two steps:



Step 1: We will show that after some non-zero number of rotations, degree measure of the arc between P and P_0 will be less than 0.001.

Step 2: Repeat step 1 many times, and we will eventually end up inside γ.

Irrational Rotations

Write down the degree measures of P from P_0 mod 360.

That is, write down the sequence α, 2α, 3α, 4α, … mod 360 in decimal form.

$$\begin{array}{rl}
\alpha = & 144.321314561\ldots \\
2\alpha = & 288.642629122\ldots \\
3\alpha = & 072.963943683\ldots \\
4\alpha = & 217.285258245\ldots \\
5\alpha = & 001.606572806\ldots \\
6\alpha = & 145.927887367\ldots \\
\vdots & \qquad \vdots
\end{array}
\longrightarrow
\begin{array}{rl}
\ldots\alpha = & 233.571314571\ldots \\
\ldots\alpha = & 289.647295125\ldots \\
\ldots\alpha = & 272.253943673\ldots \\
\ldots\alpha = & 217.285258245\ldots \\
\ldots\alpha = & 221.706279706\ldots \\
\ldots\alpha = & 275.979887367\ldots \\
\vdots & \qquad \vdots
\end{array}$$

Irrational Rotations

Write down the degree measures of P from P_0 mod 360.

That is, write down the sequence α, 2α, 3α, 4α, … mod 360 in decimal form.

$$
\begin{aligned}
\ldots\alpha &= 273.576334571\ldots\\
\ldots\alpha &= 274.544295185\ldots\\
\ldots\alpha &= 273.243946375\ldots\\
\ldots\alpha &= 277.255258245\ldots\\
\ldots\alpha &= 278.704379606\ldots\\
\ldots\alpha &= 278.579883468\ldots\\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
\ldots\alpha &= 233.571314571\ldots\\
\ldots\alpha &= 289.647295125\ldots\\
\ldots\alpha &= 272.253943673\ldots\\
\ldots\alpha &= 217.285258245\ldots\\
\ldots\alpha &= 221.706279706\ldots\\
\ldots\alpha &= 275.979887367\ldots\\
&\vdots
\end{aligned}
$$

Irrational Rotations

Write down the degree measures of P from P_0 mod 360.

That is, write down the sequence $\alpha,\ 2\alpha,\ 3\alpha,\ 4\alpha,\ \ldots$ mod 360 in decimal form.

$$
\begin{array}{ll}
\ldots\alpha = & 273.576334571\ldots \\
\ldots\alpha = & 274.544295185\ldots \\
\ldots\alpha = & 273.243946375\ldots \\
\ldots\alpha = & 277.255258245\ldots \\
\ldots\alpha = & 278.704379606\ldots \\
\ldots\alpha = & 278.579883468\ldots \\
\vdots & \qquad\vdots
\end{array}
\quad\longrightarrow\quad
\begin{array}{ll}
\ldots\alpha = & 279.176334571\ldots \\
\ldots\alpha = & 279.524234185\ldots \\
\ldots\alpha = & 279.243946334\ldots \\
\ldots\alpha = & 279.245252245\ldots \\
\ldots\alpha = & 279.104379646\ldots \\
\ldots\alpha = & 279.379183462\ldots \\
\vdots & \qquad\vdots
\end{array}
$$

Irrational Rotations

Write down the degree measures of P from P_0 mod 360.

That is, write down the sequence $\alpha,\ 2\alpha,\ 3\alpha,\ 4\alpha,\ \ldots$ mod 360 in decimal form.

$$
\begin{array}{rl}
\ldots\alpha = & 279.147534571\ldots \\
\ldots\alpha = & 279.147534185\ldots \\
\ldots\alpha = & 279.147546334\ldots \\
\ldots\alpha = & 279.147552245\ldots \\
\ldots\alpha = & 279.147579646\ldots \\
\ldots\alpha = & 279.147583462\ldots \\
\vdots & \qquad\quad\ \vdots
\end{array}
$$

After 4 more times

$$
\begin{array}{rl}
\ldots\alpha = & 279.176334571\ldots \\
\ldots\alpha = & 279.524234185\ldots \\
\ldots\alpha = & 279.243946334\ldots \\
\ldots\alpha = & 279.245252245\ldots \\
\ldots\alpha = & 279.104379646\ldots \\
\ldots\alpha = & 279.379183462\ldots \\
\vdots & \qquad\quad\ \vdots
\end{array}
$$

Irrational Rotations

Write down the degree measures of P from P_0 mod 360.

That is, write down the sequence α, 2α, 3α, 4α, … mod 360 in decimal form.

$$
\begin{aligned}
X\alpha &= 279.147534571\ldots \\
Y\alpha &= 279.147534185\ldots \\
\ldots\,\alpha &= 279.147546334\ldots \\
\ldots\,\alpha &= 279.147552245\ldots \\
\ldots\,\alpha &= 279.147579646\ldots \\
\ldots\,\alpha &= 279.147583462\ldots \\
&\vdots
\end{aligned}
$$

By applying infinite pigeonhole principle seven times, we can find positive integers X and Y such that

- $X < Y$
- $X\alpha$ and $Y\alpha$ are equal up to the 4$^{\text{th}}$ decimal place.

Then, $|(Y - X)\alpha| < 0.001.$

So, after rotating $Y - X$ times, measure of arc PP_0 is less than 0.001 degrees!!!

That's it for this lecture.

See you tomorrow!