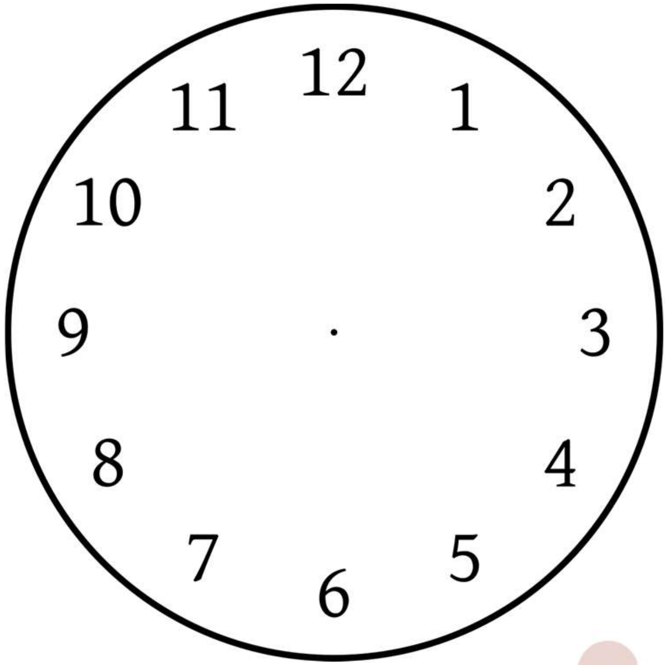
We will begin at 8:35 MMT

Try this problem while we wait…

Draw a straight line through the surface of the clock given to the right so that sum of the numbers on each side of the line are equal.

If you replace $1, 2, \ldots, 12$ with $1, 2, \ldots, n$, for which n can you draw such a line?



Record the meeting…

Some Housekeeping

- Diamond problems for Problem Set 3 will be due tomorrow. Please submit by personal contact only.

- We already reached half-way point of the training. I hope you are half-way prepared for the IMO (although the nature of improvement may not be linear).

Content so far…

Lecture – 9

Injections and Bijections

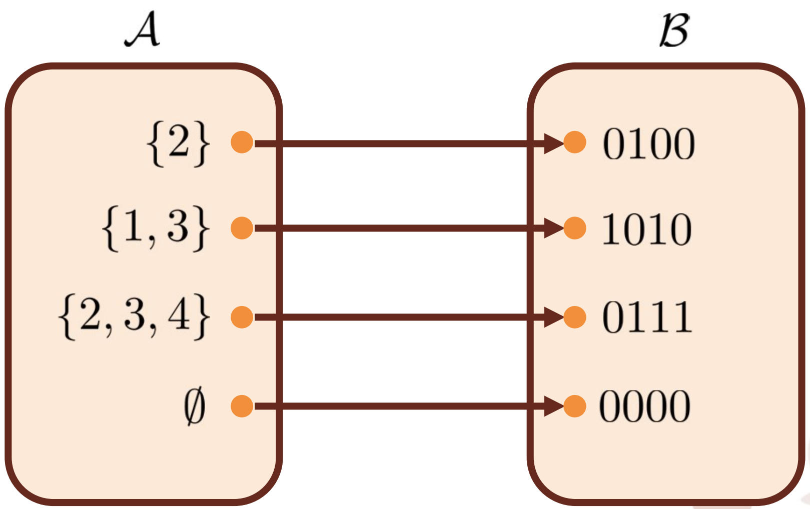
Subsets vs. Bit-strings

Consider the following two collections:

- Collection of all subsets of $\{1, 2, 3, \ldots, n\}$. ← Call this collection $\mathcal{A}$
- Collection of strings of length n consisting of 0s and 1s only (called bit-strings of length n). ← Call this collection $\mathcal{B}$

Well, they have the same size. Why?

Some examples for $n = 4$



Consider the function from $\mathcal{A}$ to $\mathcal{B}$ taking each subset S into a bit-string of length n as follows:

The k-th entry is 1 if $k \in S$.
The k-th entry is 0 if $k \notin S$.

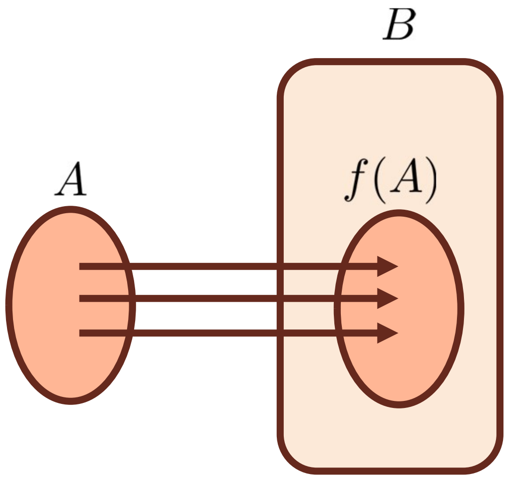
Equal to 2^n.

This function is bijection! So, $|\mathcal{A}| = |\mathcal{B}|$.

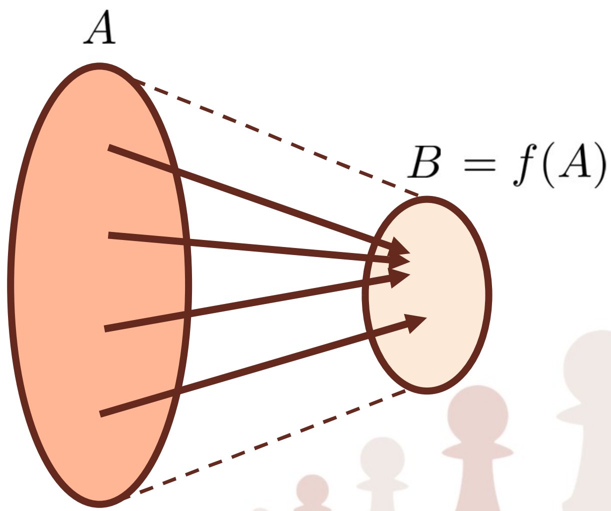
Injective, Surjective and Bijective

Let A and B be two finite sets and $f : A \to B$ be a function.

- Injective: If $f(a) = f(b)$ implies $a = b$.
  - Combinatorial view: given an element in the range, we can uniquely recover its input.
  - Consequence: $|A| \leq |B|$.



- Surjective: For each $y \in B$, there is $x \in A$ such that $f(x) = y$.
  - Combinatorial view: we can create any output we wish.
  - Consequence: $|A| \geq |B|$.

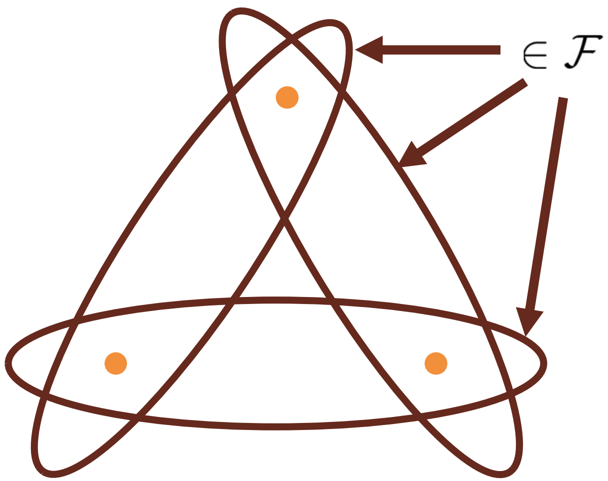


- Bijective: Both injective and surjective. ← If and only if f^{-1} exists.
  - Consequence: $|A| = |B|$.

Intersecting Families

Let $\mathcal{F}$ be the collection of (some) subsets of $\{1, 2, 3, \ldots, n\}$ such that any two sets in $\mathcal{F}$ have non-empty intersection. What is the maximum size of $|\mathcal{F}|$?

Bit-string version: Let $\mathcal{F}$ be a set containing bit-strings of length n such that for any two strings s and t in $\mathcal{F}$, there is a positive integer $1 \leq k \leq n$ such that the k-th bit of s and t is 1.



(OR) $\mathcal{F} = \{110, 011, 101\}$

Small Examples

- For $n = 2$, we have $\emptyset$, $\{1\}$, $\{2\}$, $\{1, 2\}$
  - We can't put $\emptyset$ in, and we should put $\{1, 2\}$ in because it will never cause any problem. We can't have $\{1\}$ and $\{2\}$ in simultaneously.
  - So, max. we can select is 2.

- For $n = 3$, we have $\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$
  - We can't put $\emptyset$ in, and we should put $\{1, 2, 3\}$ in. We can't have the following pairs simultaneously.

$$\{1\} \leftrightarrow \{2, 3\} \qquad \{2\} \leftrightarrow \{1, 3\} \qquad \{3\} \leftrightarrow \{1, 2\}$$

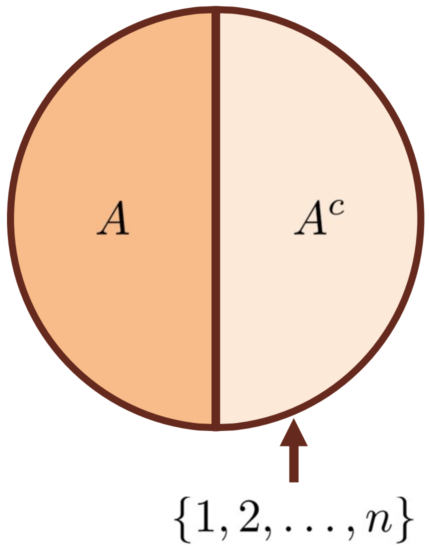
  - So, we cannot select more than 4.
  - Can we select exactly 4? ← Yes.

Part One: Cannot be more than …

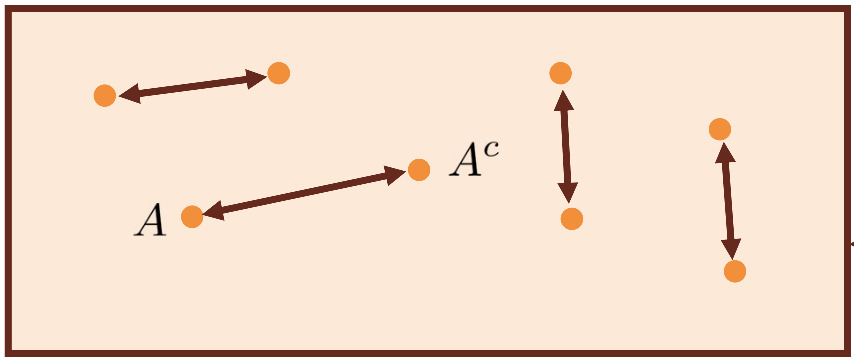
- Idea: If $\mathcal{F}$ contains a set A, is there another set that $\mathcal{F}$ cannot contain?

- Answer: Yes. Complement of A.

- But, A and complement of A form a pairing in the set of all possible subsets. Thus, $\mathcal{F}$ cannot have more than half of all possible subsets.



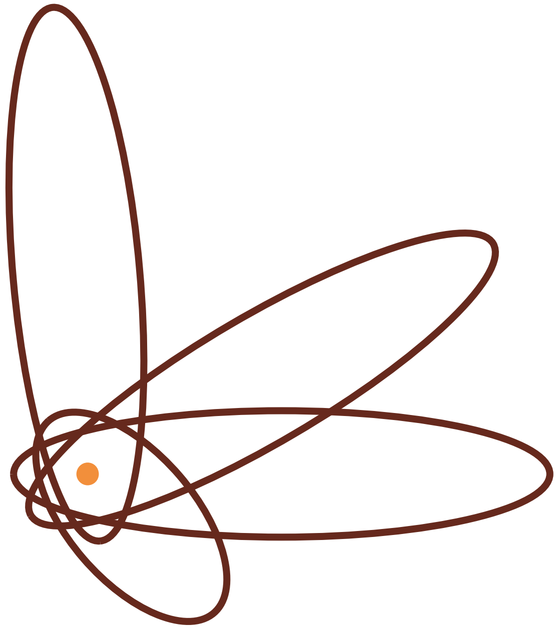
This shows that max size $\leq 2^{n-1}$



Collection of all subsets of $\{1, 2, \ldots, n\}$

Part Two: Can construct …

- Let us think of a way to construct $\mathcal{F}$ so that it contains a lot of sets and satisfies the desired property.
- Observation: If all the sets in $\mathcal{F}$ have a common element, then the desired property is satisfied.
- So, construct $\mathcal{F}$ by choosing all the subsets that contain 1.
- How many sets does $\mathcal{F}$ contain? ⟵ 2^{n-1}
- So, the maximum possible size of $\mathcal{F}$ is 2^{n-1}.

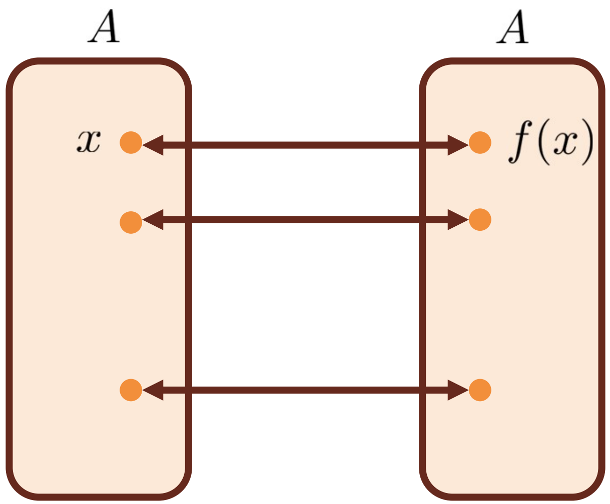


Involutions Induce Pairings

Let A be a set. Consider a function $f : A \to A$ such that $f(f(x)) = x$ for all $x \in A$. Such function f is called an involution.

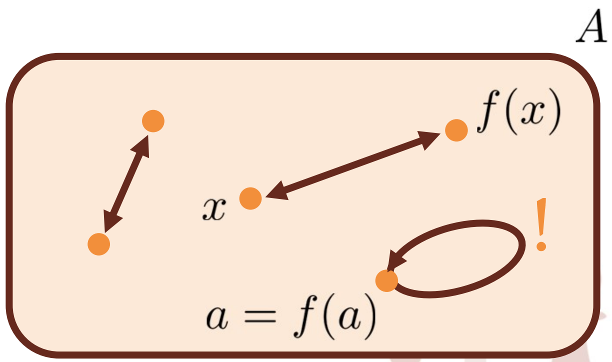
This implies that $f^{-1} = f$.

In particular, f is bijective.



Examples

- $A =$ set of all subsets of $\{1, \ldots, n\}$. f is ''taking complement''.
- $A =$ set of all bit-strings of length n. f is "Changing the first bit from 0 to 1 or from 1 to 0".
- $A =$ set of all positive reals. f is "taking reciprocal".



Bijections vs. Alternating Sums

Let $n > 100$ be an integer. Let's try to find the closed form of the sum:

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots - \binom{n}{99} + \binom{n}{100}.$$

As always, let's try small examples: replace 100 with 4 and n with 6. ← Tip: it is usually good to keep the parity in the problem.

Positive terms: $\binom{6}{0} + \binom{6}{2} + \binom{6}{4}$ ← Counts even subsets of size ≤ 4.

Negative terms: $\binom{6}{1} + \binom{6}{3}$ ← Counts odd subsets of size ≤ 4.

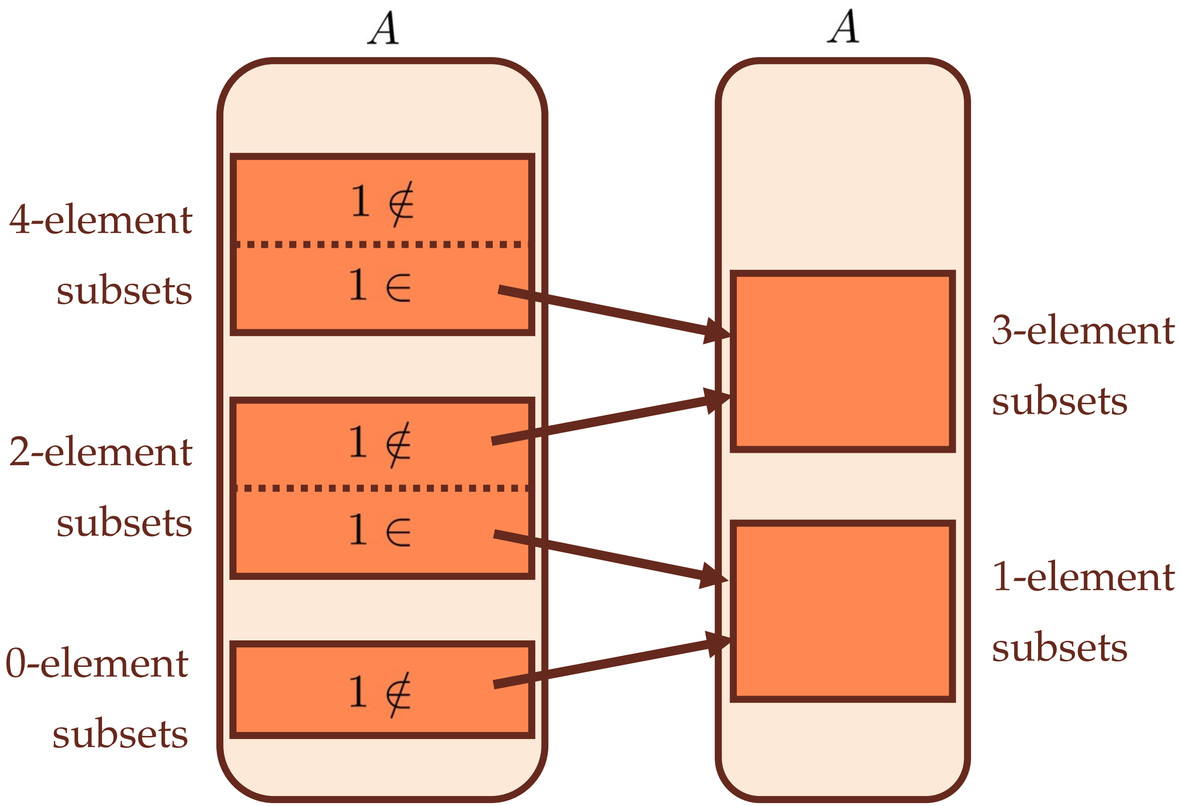
So, we should "pair up" even and odd subsets of size ≤ 4 so that we cancel out many terms.

Pairing Up

Let A be the set of all subsets of $\{1, 2, 3, \ldots, 6\}$. Consider the involution: flipping the first bit.

If $X \in A$ contains 1, remove it.

If $X \in A$ doesn't contain 1, put it in.



By the picture to the left, we have

$$\binom{6}{0} + \binom{6}{2} + \binom{6}{4} - \binom{6}{1} - \binom{6}{3}$$

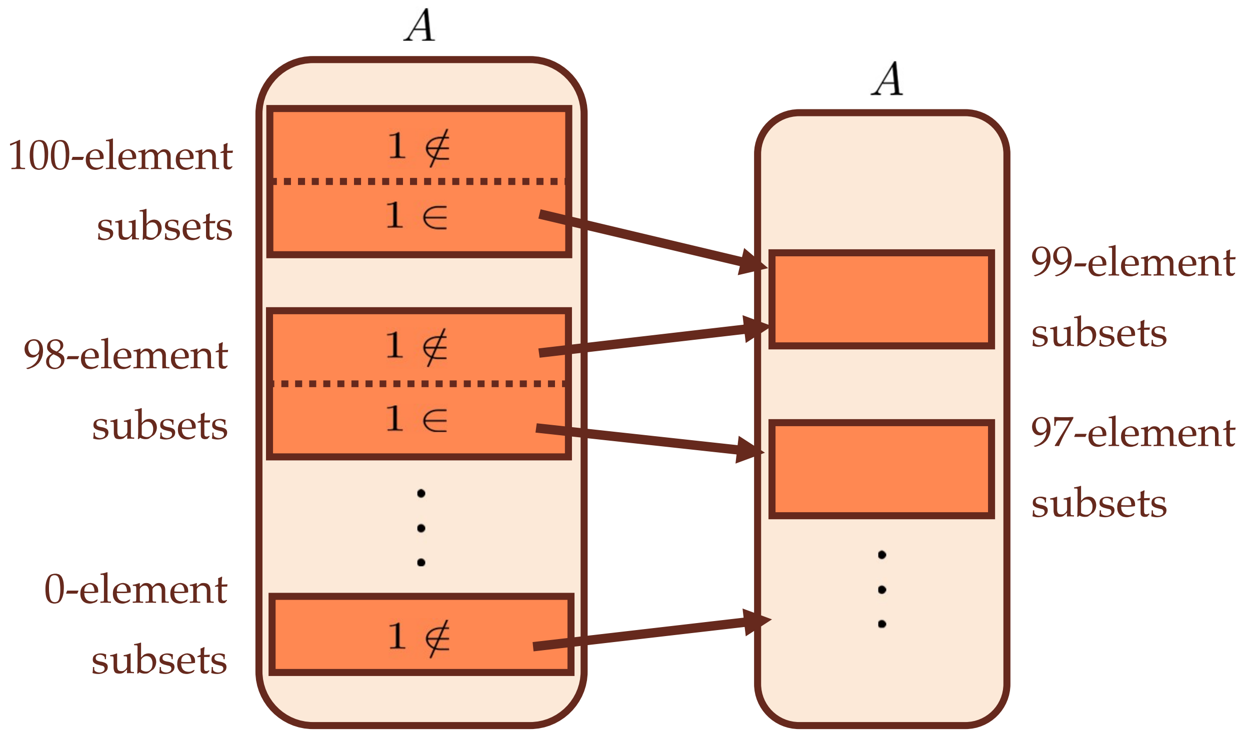
$=$ number of 4-element subsets that doesn't contain 1

$= \binom{5}{4}$ ← Answer

General Case

We wish to evaluate: $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots - \binom{n}{99} + \binom{n}{100}$.

Let A be the set of all subsets of $\{1, 2, 3, \ldots, n\}$. Consider the involution: flipping the first bit.



After cancellation, we are left with the number of 100-element subsets that does not contain 1.

So, the answer is $\binom{n-1}{100}$.

What bijections enable us to do

Let A and B be two finite sets.

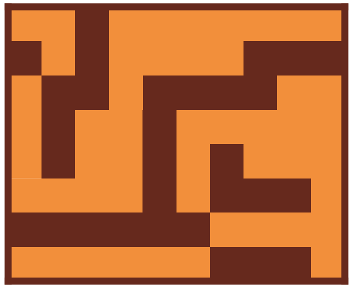
- If there is a bijection $f : A \to B$, then A and B have the same size.
- Via f, we can change our point of view on elements of A by thinking them as elements of B. This change in perspective sometimes clarify things up.

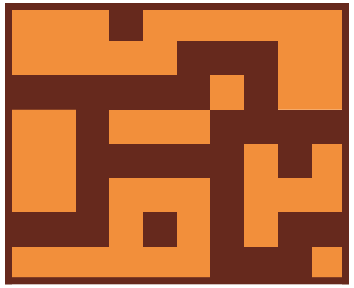
These are very useful if A and B are seemingly unrelated.

- If $A = B$ and f is an involution, then f induces a pairing of elements in A. This allows us to do fancy arguments: pigeonhole on pairs, alternating sums, etc.

Crossing the Board (IMOSL 2005/C3)

Consider an $m \times n$ rectangular board consisting of mn unit squares. Two of its squares are called adjacent if they have a common edge, and a path is a sequence of unit squares in which any two consecutive squares are adjacent. Two paths are called non-intersecting if they do not share any common squares. A colouring of the board is an assignment of a colour: brown or orange to each of the mn unit squares.





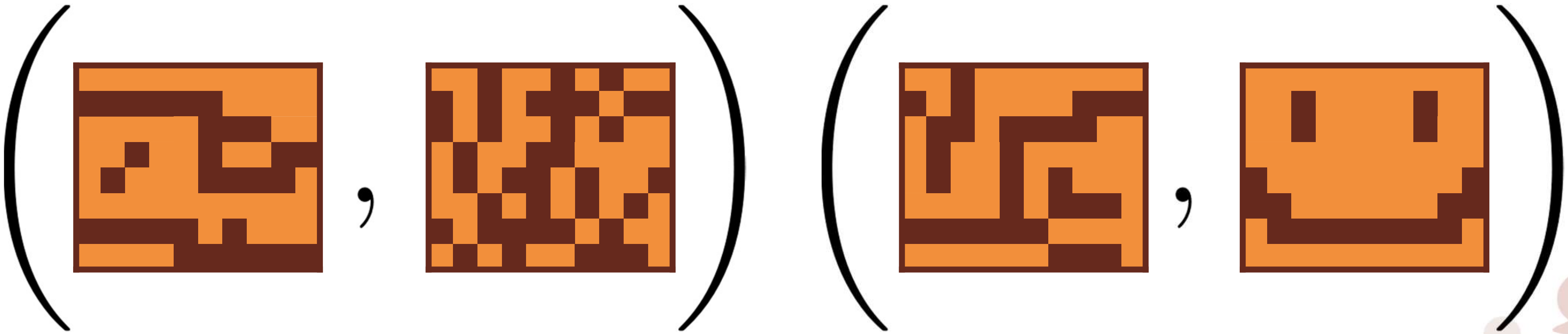
Let N be the number of colourings of the board such that there exists at least one brown path from left edge of the board to its right edge. Let M be the number of colourings of the board for which there exist at least two non-intersecting brown paths from the left edge of the board to its right edge. Prove that $N^2 \geq M \cdot 2^{mn}$.

The Setup

Construct two sets A and B with $|A| = M \cdot 2^{mn}$ and $|B| = N^2$. Then, setup an injective map.

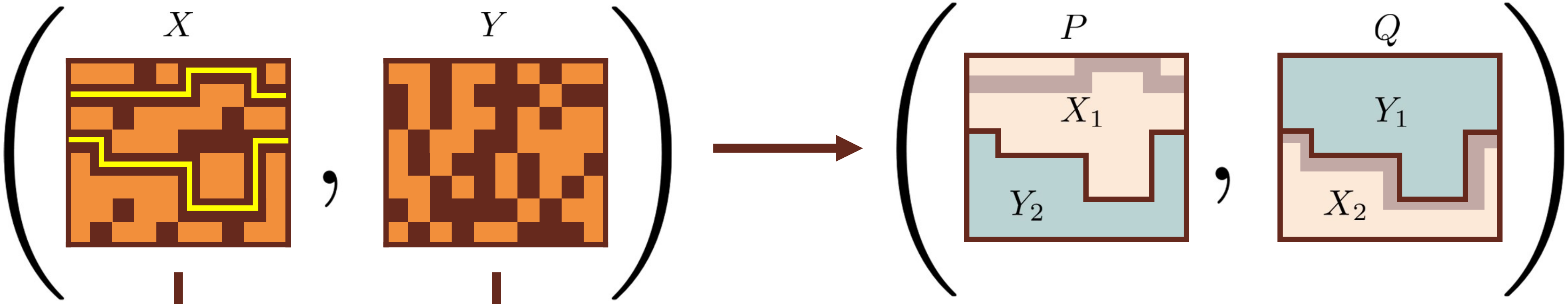
- $A =$ set of pairs of colourings X and Y where X has 2 disjoint L-to-R brown paths, and Y is an arbitrary colouring.

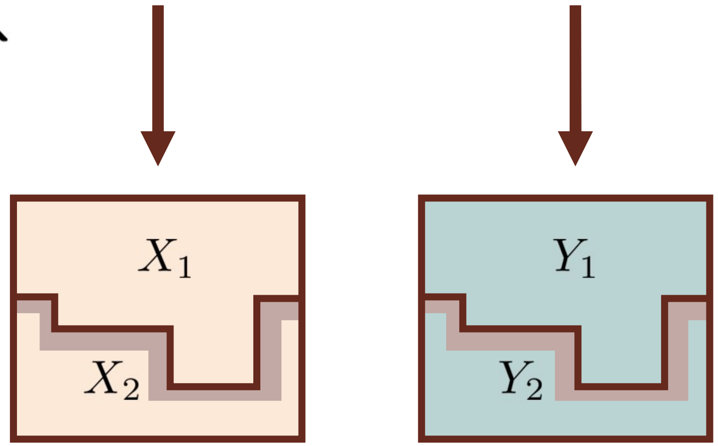
- $B =$ set of pairs of colourings P and Q where both P and Q have an L-to-R brown path.



Constructing the Function

We want the mapping to be injective. That is, we must be able to uniquely recover the input from a given output.

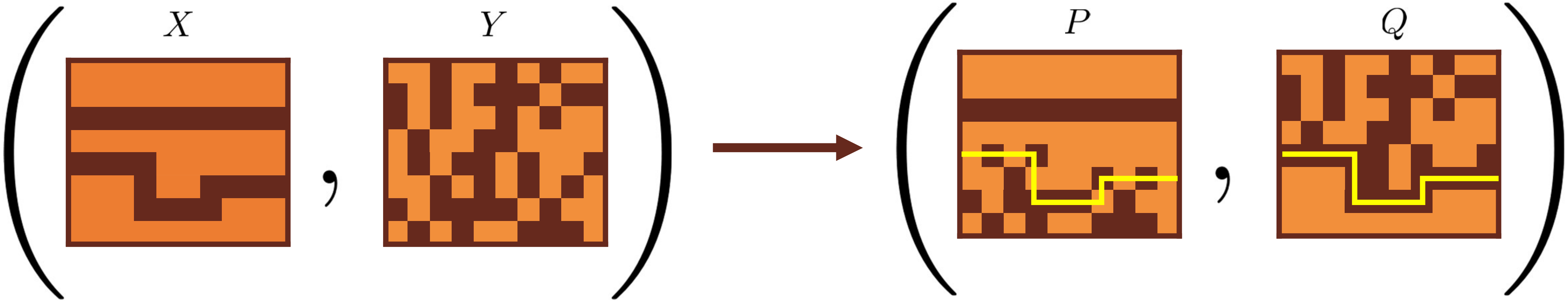




Take a brown path in X that minimizes bottom area.

Use this path to cut X and Y into two parts each: the part above the path and the remaining part. Then, shuffle the parts.

Reconstructing the Input from the Image



Given an output, look at the brown path in Q that minimizes the bottom area.

Now, you can cut P and Q into two parts each: the part above the path and the remaining part. Reshuffling recovers the input uniquely.

Therefore, there is only one pair (X, Y) such that $f(X, Y) = (P, Q)$ i.e. our f is injective.

That's it for this lecture.

See you tomorrow!