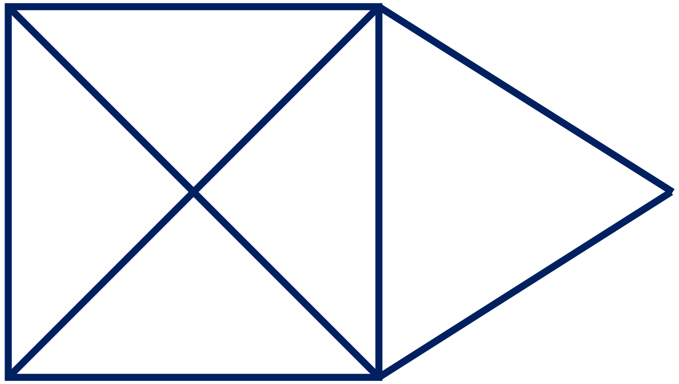
We will begin at 1:05 PM MMT

Try this puzzle while we wait…

Try to draw this picture on your paper without lifting your pen and without tracing any segment more than once. Can you start from anywhere? If not, which starting points are not valid?



Record the meeting…

Some Housekeeping

I'll try a new approach to this week's problem solving session. So, please join the new Zoom meeting link after this lecture (will be sent in this Zoom chat as well as messenger chat groups).

Content so far…

L1: Monovariants L2: Invariants L3: Alternating-variants	I
L4: Inductive constructions L5: Greedy and RUST	II
L6: Counting in two ways L7: (Bonus) Polyhedron formula L8: (Bonus) Counting in graphs L9: Injections and bijections	III
L10: Pigeonhole principle L11: Continuity and descent L12: Leveraging symmetry	IV
L13: Combinatorial games	V
L14: Combinatorial geometry	VI
L15: Results in graph theory I L16: Results in graph theory II	VII

Lecture – 8

Counting in Graphs

Some Special Graphs

Simple finite graphs in which any two vertices are connected by an edge are called complete graphs.

These are usually denoted by the symbol K_n where n is the number of vertices.

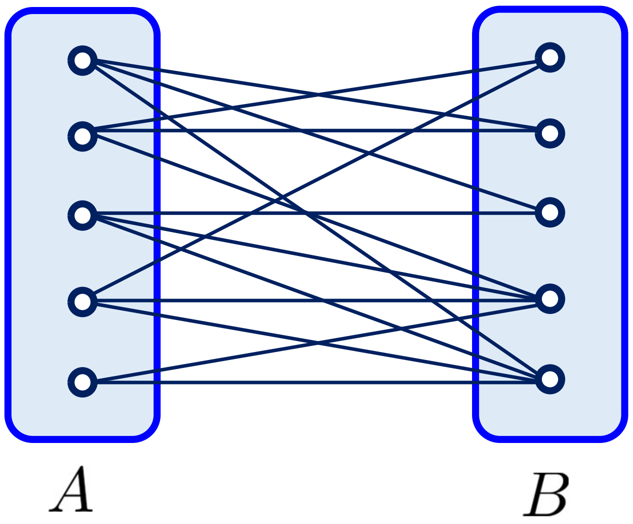
This will be assumed throughout this course unless otherwise stated.

K_1 K_2 K_3 K_4 K_5

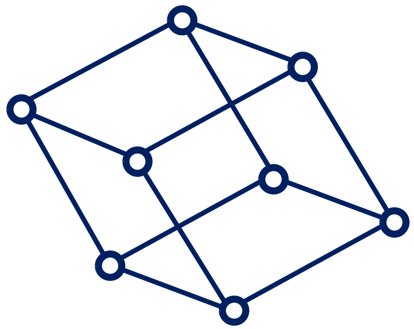
Number of edges in $K_n = \dfrac{n(n-1)}{2} \longleftarrow \sim \dfrac{n^2}{2}$

Some Special Graphs

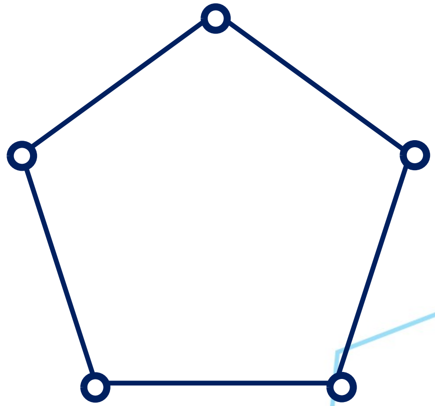
Suppose that the vertex set V of a graph can be written as disjoint union of two sets A and B such that there is no edge in A and that there is no edge in B. Such a graph is called bipartite.



Bipartite



Bipartite

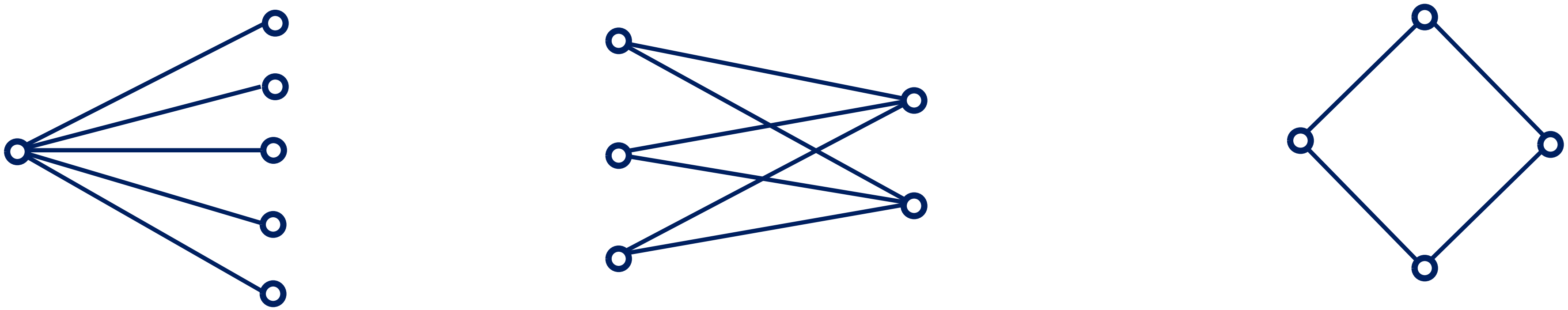


NOT Bipartite

Note. Obviously, bipartite graphs do not contain odd-cycles (i.e. cycles of odd length). The converse is also true (PSet2 Further Exercise).

Some Special Graphs

If every edge between A and B is present, we call it a complete bipartite graph. We denote them by the symbol $K_{s,t}$ where $s = |A|$ and $t = |B|$.



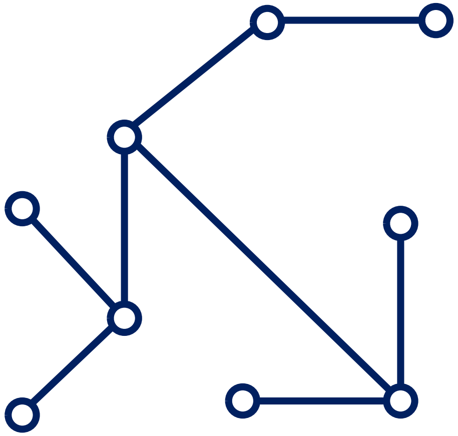
$K_{1,5}$ $K_{3,2}$ $K_{2,2}$

Number of edges in $K_{s,t} = st$

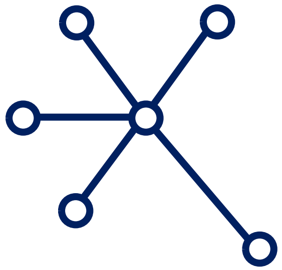
Maximum number of edges in $K_{s,t}$ is $\lfloor n^2/4 \rfloor$ by AM-GM. $\longleftarrow$ $st \leq \left(\frac{s+t}{2}\right)^2 = \frac{n^2}{4}$

Some Special Graphs

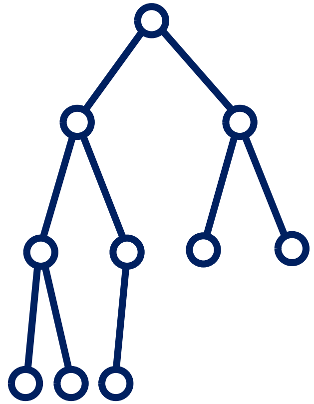
A connected graph with no cycles is called a tree.



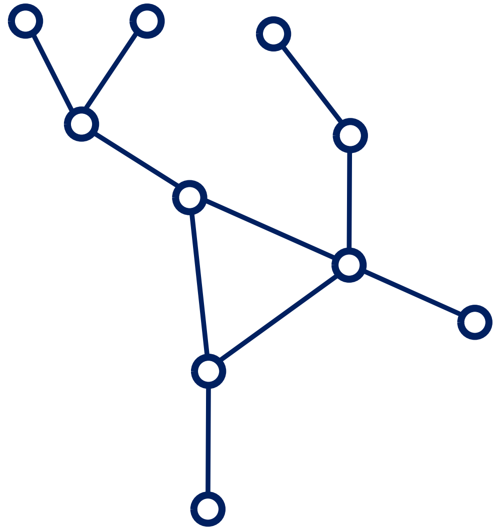
Tree



Tree



Tree



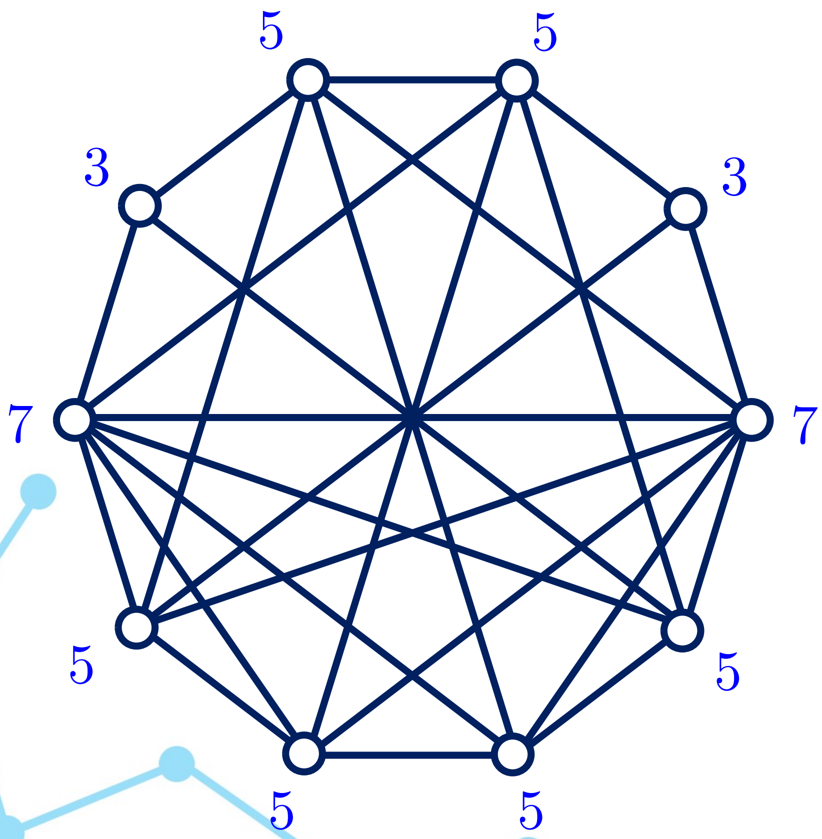
NOT Tree

Fact. A tree with n vertices has $n - 1$ edges.

Proof Sketch: Every tree has a vertex of degree 1 (for example, the ending vertex of the longest path). Remove it and use induction.

The Most Important Identity

How many edges are there in the following graph?



It is kind of tricky to count with your eye, there are just too many lines.

Trick: Just count how many edges are joined to each vertex.

The answer is –

$$\frac{3+3+7+7+5+5+5+5+5+5}{2} = 25$$

Each edge is counted twice in the numerator.

The Most Important Identity

The number of edges incident to a vertex v is called its degree, and here we will write $\deg(v)$.

Theorem. Let G be any (possibly not simple) graph with vertex set V and edge set E. Then,

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Quick Question: Can you draw a graph with 6 vertices whose degrees are 2, 2, 2, 3, 4, 4?

No, sum of degrees should be even.

Corollary. In any (possibly not simple) graph, number of odd-degree vertices is even.

Consequences of the Main Identity

Let G be a graph with vertex set V and edge set E. Let $|V| = n$ and $|E| = m$. Then,

$$\sum_{v \in V} \deg(v) = 2m, \longleftarrow \text{The main identity}$$

$$\sum_{v \in V} \deg(v)^2 \geq \frac{4m^2}{n}, \longleftarrow \text{Main identity + AM-QM}$$

$$\sum_{v \in V} \binom{\deg(v)}{2} \geq \frac{2m^2}{n} - m, \longleftarrow \text{Follows from above}$$

Note: Just remember how to derive them. No need to memorize except the main identity.

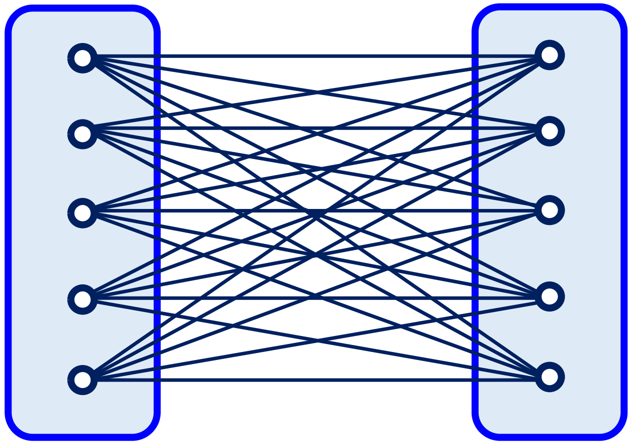
The TST Problem

There are 10 points on a plane. Any three of them are not collinear. There are at least 26 line segments joining pairs of the points. Show that there are three line segments forming a triangle (whose vertices belong to the original 10 points).

Graph Formulation:

Let G be a (simple) graph on 10 vertices. If G has at least 26 edges, show that G contains a triangle.

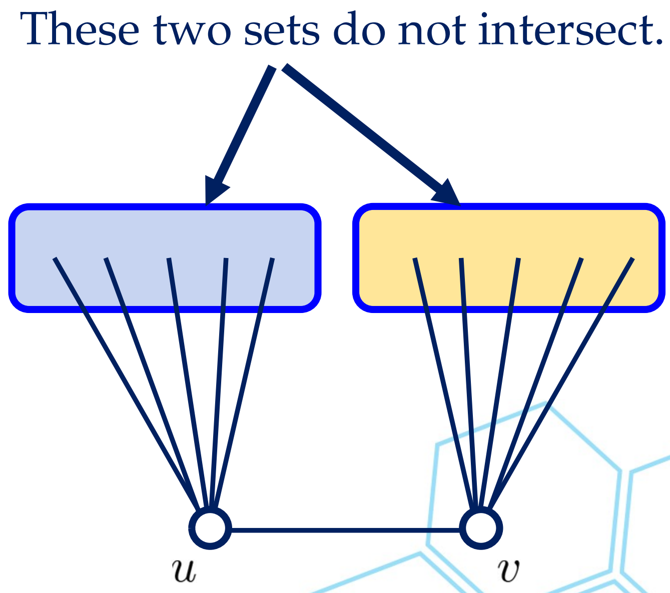
Why 26? Well, there is a graph with 25 edges and have no triangles.



Mantel's Theorem

Theorem. Let G be a (simple) graph with no triangles. Then, G contains at most $n^2/4$ edges.

Proof: Suppose that G contains m edges.



- For any edge $uv \in E$, we have

$$\deg(u) + \deg(v) \leq n.$$

- Adding up over all possible edges $uv \in E$, we get

$$\sum_{uv \in E} (\deg(u) + \deg(v)) \leq mn.$$

- But, LHS is equal to $\sum_{v \in V} \deg(v)^2$. Hence, by useful fact 2:

$$\frac{4m^2}{n} \leq mn \longrightarrow \boxed{m \leq \frac{n^2}{4}}$$

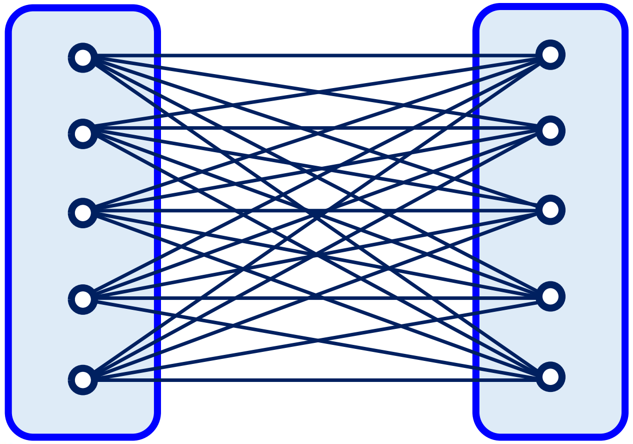
Mantel's Theorem

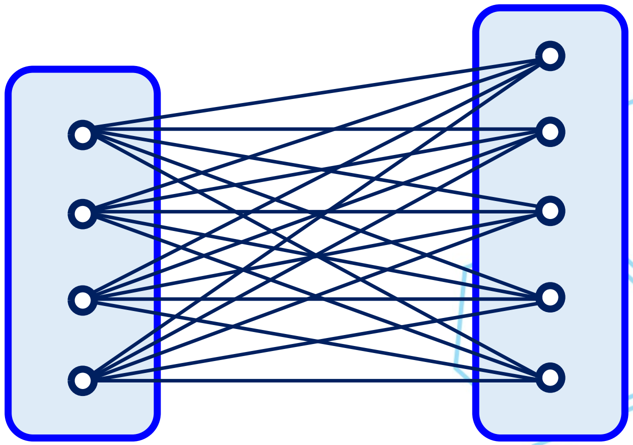
The number $n^2/4$ is best possible.

There is a triangle-free graph with exactly $\lfloor n^2/4 \rfloor$ edges:

That is, the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$

Separate the vertices into two sets of sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$.
Join every "crossing" edges.

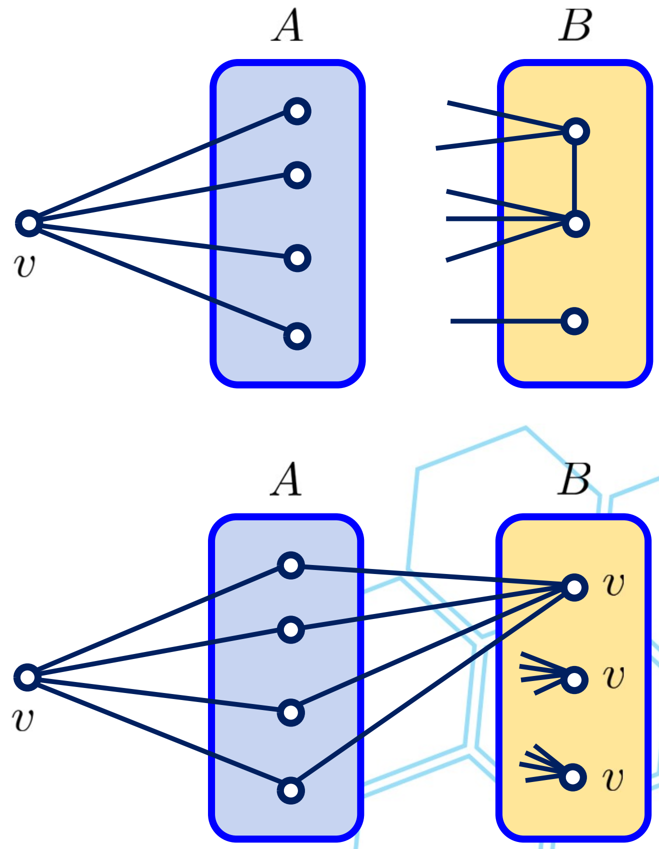




Non-counting Proof

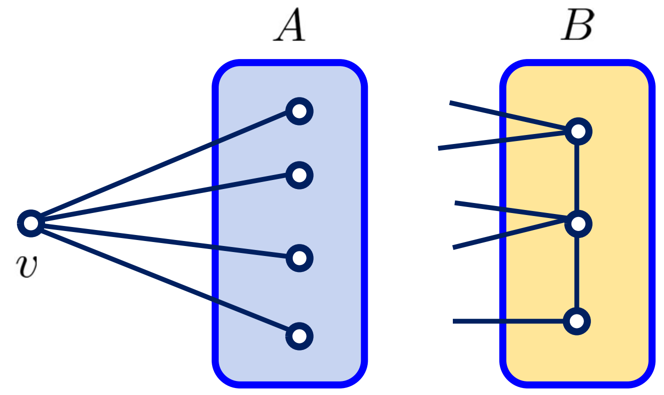
We will show that given a triangle-free graph G, we can change the edge set of G so that the resulting graph is bipartite and degree of each vertex does not decrease.

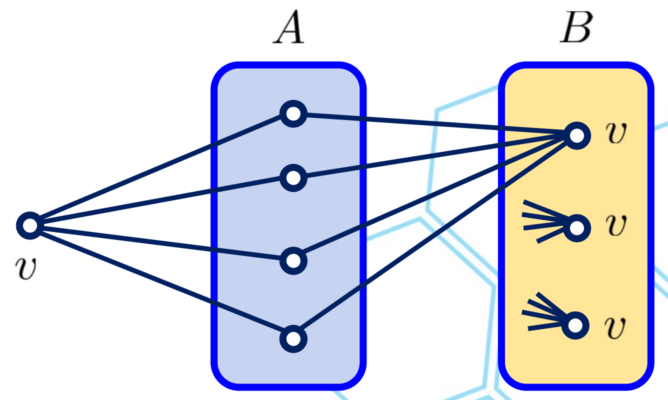
- Look at the vertex v with largest degree.
- Let A be neighbors of v and B be non-neighbors of v.
- Observation: No vertex in A is joined to each other. Every vertex in B has degree at most that of v.
- Main Idea: Simply replace every vertex in B with a clone of v.



Non-counting Proof

- **Observation:** No vertex in A is joined to each other. Every vertex in B has degree at most that of v.
- **Main Idea:** Simply replace every vertex in B with a clone of v.
- Then, the resulting graph is bipartite: neighbors of v versus clones of v together with v himself.
- Also, the degrees do not decrease. Hence, number of edges also does not decrease.
- So, we are done because number of edges in a bipartite graph is at most $\lfloor n^2/4 \rfloor$.





Counting vs. Non-counting

Global Arguments

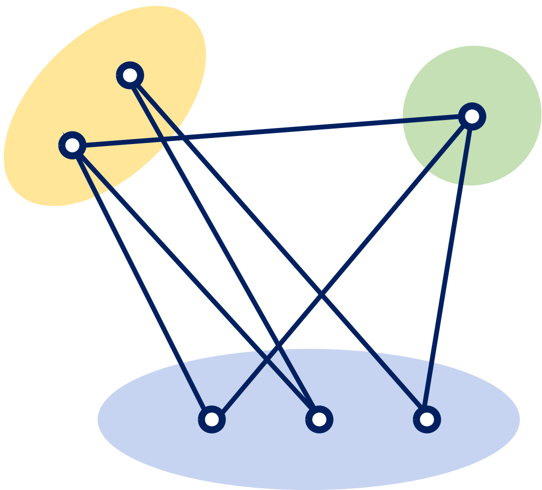
- Easier to setup: just find an inequality/equality that you can add up. So, you usually have something to do without having to stare at the problem.
- Computations can get very messy.
- They are sometimes not enlightening.

Non-global Arguments

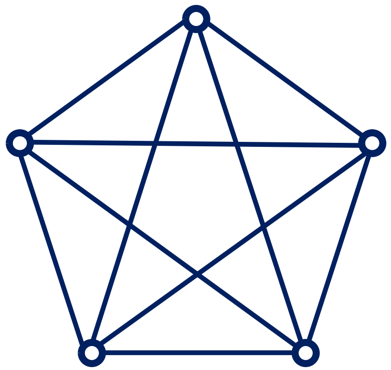
- Much more difficult to come up with.
- Clean, nice and beautiful once it is understood.
- Usually very enlightening and explains what is going on behind the scene.

r-partite Graphs

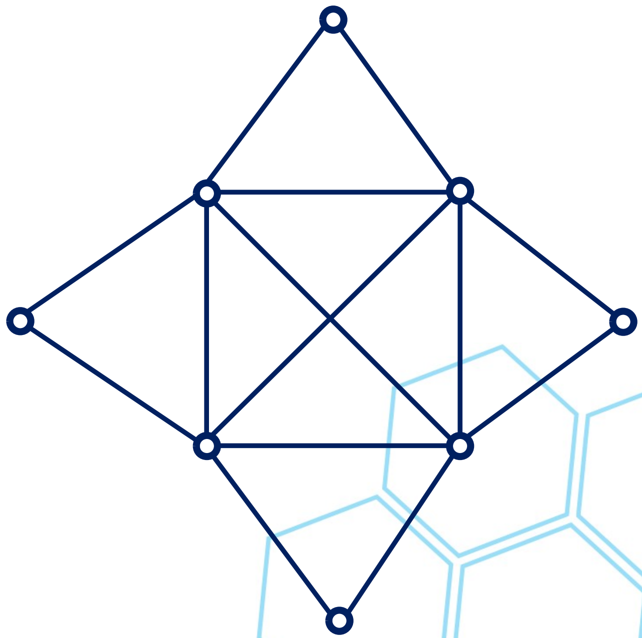
If the vertex set V is a graph can be written as $V = A_1 \cup A_2 \cup \cdots \cup A_r$ where $A_1, A_2, \ldots, A_r$ are pairwise disjoint and there is no edge between each individual A_i , then we say that the graph is r-partite.



3-partite graph



5-partite graph



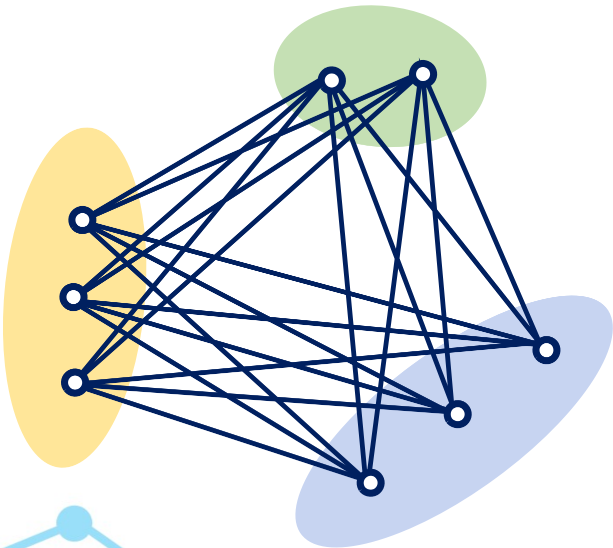
4-partite graph

r-partite Graphs

What is the maximum number of edges in an r-partite graph with n vertices?

Intuition:

We should join every crossing edge. We should have about equal number of vertices in each A_i .



More precisely, we want the difference between sizes of A_i and A_j to be at most 1.

r-partite Graphs

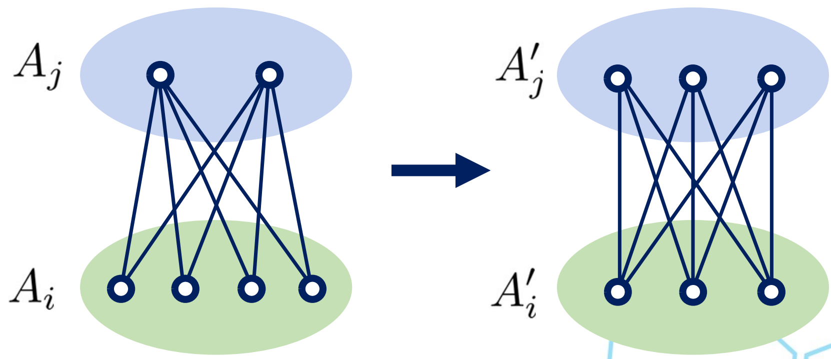
RUST: Whenever you find A_i and A_j such that $|A_i| \geq |A_j| + 2$, move one vertex from A_i into A_j.

Whenever we do this, the change in number of edges is

$$-|A_j| + (|A_i| - 1) \geq 1.$$

So, number of edges will increase with each move.

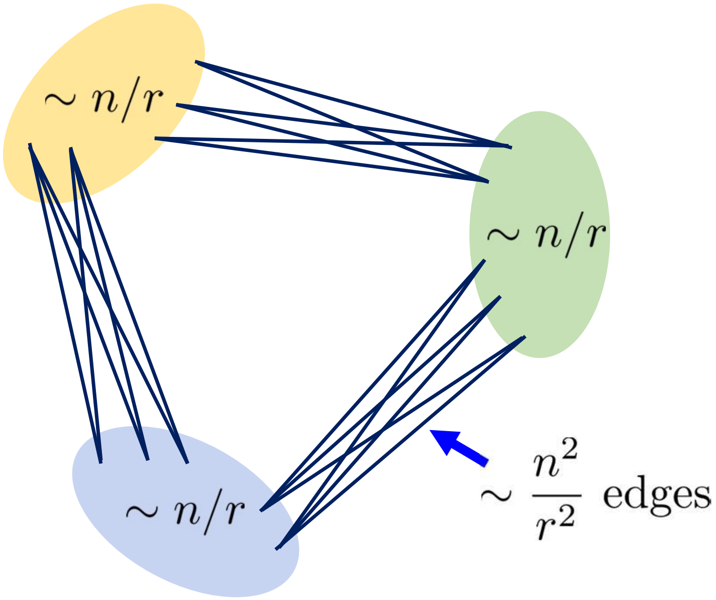
Therefore, r-partite graphs with maximum number of edges are precisely those in which every part has the about the same size.



Edges Lost : $|A_j|$
Edges Gained : $|A_i| - 1$

r-partite Graphs

So, with enough patience, we can find the maximum number of edges in the r-partite graph of n vertices. But, it is quite important to know the approximate maximum number of edges.



Max. number of edges in r-partite graph

$$\sim \binom{r}{2} \cdot \frac{n^2}{r^2}$$

$$= \frac{r-1}{r} \cdot \boxed{\frac{n^2}{2}}$$

Max. number of edges in a graph

Turán's Theorem

Theorem. Let G be a graph with that does not contain K_{r+1} as a subgraph. Then, maximum number of edges that G can have is the same as the maximum number of edges that an r-partite graph can have.

Proof. Non-counting proof of Mantel's theorem plus induction does the job. Will show in detail later.

That’s it for this lecture!

See you on Problem Solving Session .