We will begin at 4:30 PM

Try to solve the following puzzle while we wait…

In Planartopia, there are three houses and three utilities: electricity, gas and water. You want to connect each utility to each of the houses via pipes so that the pipes do not intersect with each other. Can you do it?

Also try solving this problem on a ball and on a coffee mug instead of a plane.

Some Housekeeping

- Diamond-problems for Problem Set 2 are due tomorrow 3:00 PM. No late submissions allowed. Please submit by personal contact.

- Since H_4 is now optional, I want to go over the material that are NOT problem-solving focused this week, both for an interesting journey and a show-off on how much we can do with techniques we have encountered. So, I changed titles and material of L7 and L8. I hope you don't mind. :)

- This week's problem solving session and optional problem set will review the techniques that we used in L1 to L6.

Content so far…

L1: Monovariants
L2: Invariants
L3: Alternating-variants
I

L4: Inductive constructions
L5: Greedy and RUST
II

L6: Counting in two ways
L7: (Bonus) Polyhedron formula
L8: (Bonus) Counting in graphs
L9: Injections and bijections
III

L10: Pigeonhole principle
L11: Continuity and descent
L12: Leveraging symmetry
IV

L13: Combinatorial games
V

L14: Combinatorial geometry
VI

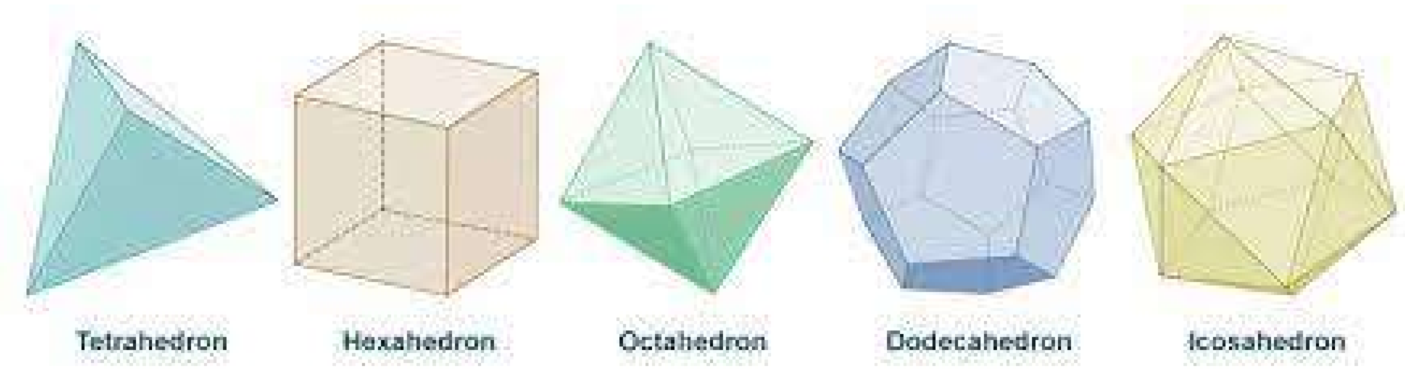
L15: Results in graph theory I
L16: Results in graph theory II
VII

Record the meeting…

Lecture – 7

The Polyhedron Formula

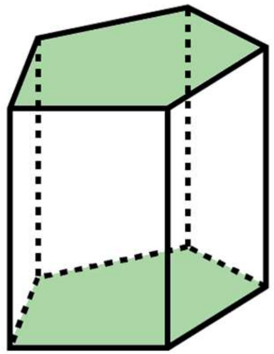
A Hidden Surprise



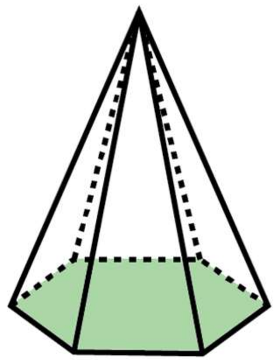
Number of vertices ($\#V$)	4	8	6	20	12
Number of vertices ($\#E$)	6	12	12	30	30
Number of vertices ($\#F$)	4	6	8	12	20
$\#V - \#E + \#F =$	2	2	2	2	2

A Hidden Surprise

Let's try to do the same for other polyhedra...



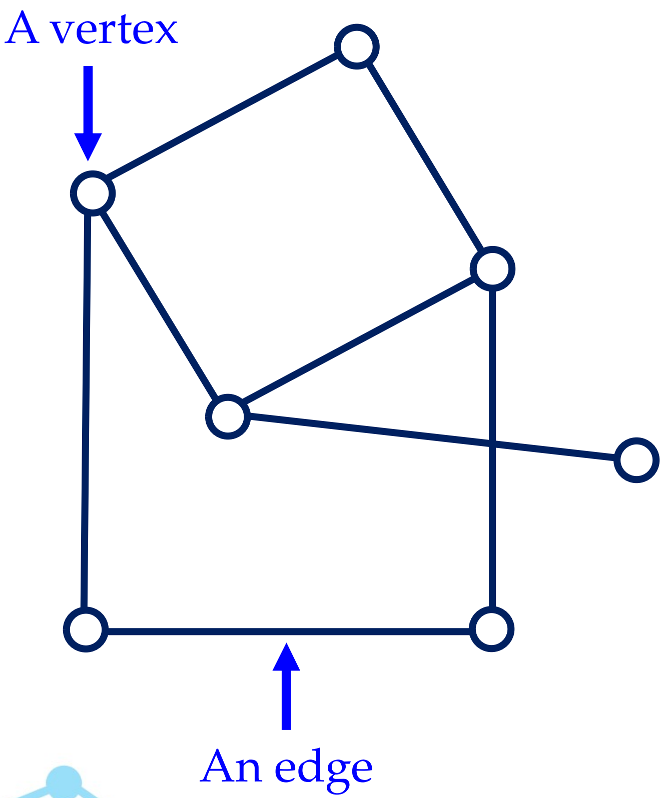
$$\#V - \#E + \#F = 10 - 15 + 7$$



$$\#V - \#E + \#F = 7 - 12 + 7$$

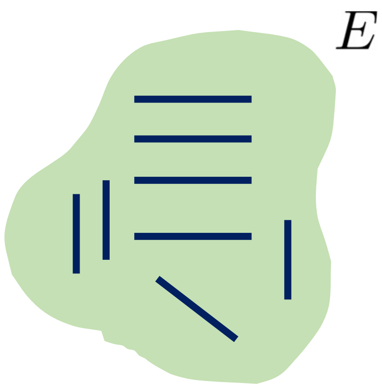
Seems like yes... $\#V - \#E + \#F = 2$ for any polyhedra.

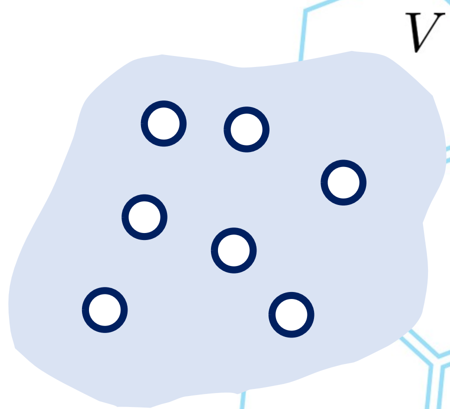
But Graph Terms First…



Formally, a graph G is a triplet (V, E, φ) where

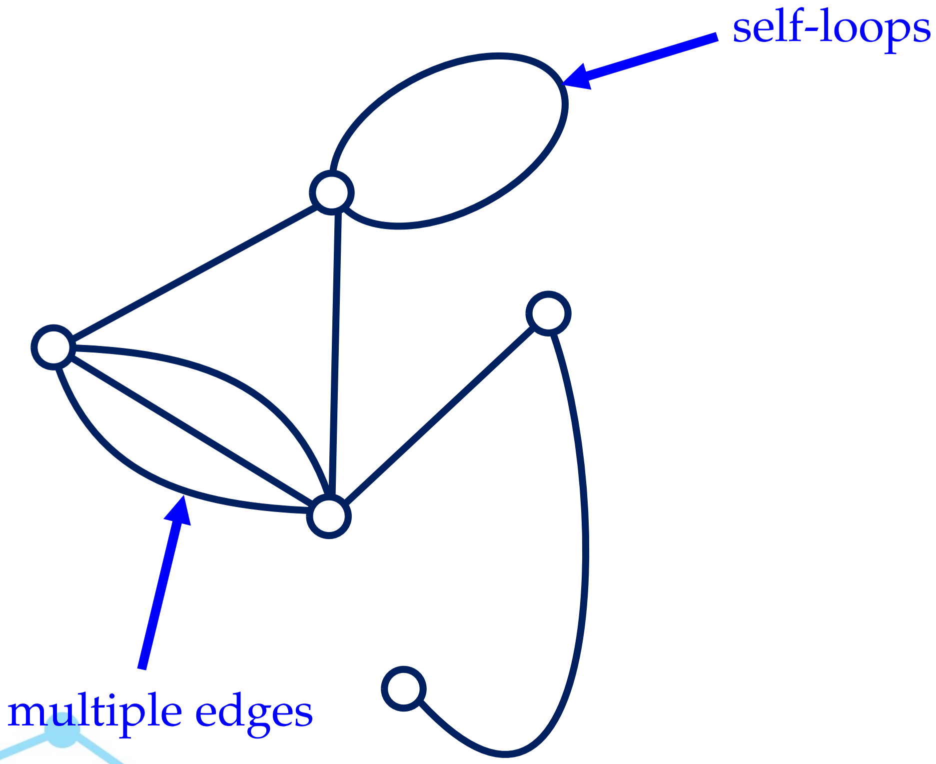
- V is a set. We call its members vertices.
- E is also a set. We call its members edges.
- φ is a function from E to $V \times V$ i.e. it assigns a pair of vertices to each edge.



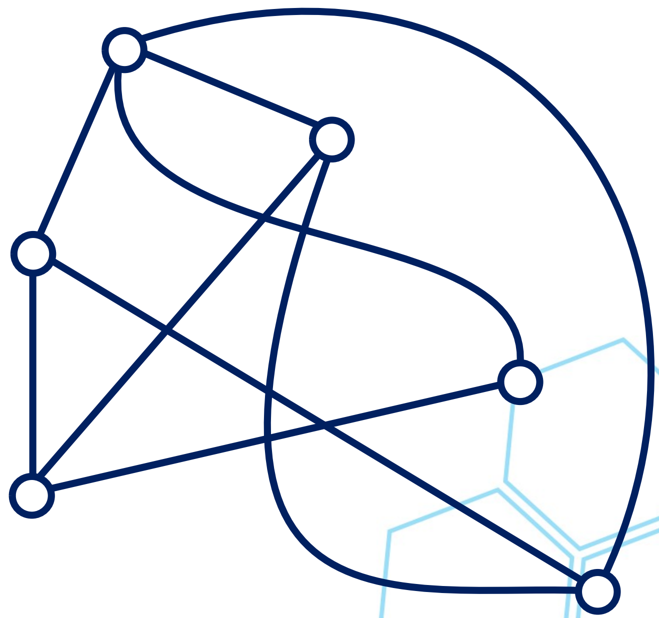


More Graph Terminology

A graph without multiple edges and self-loops are called simple. In this class, all graphs are simple unless otherwise stated.



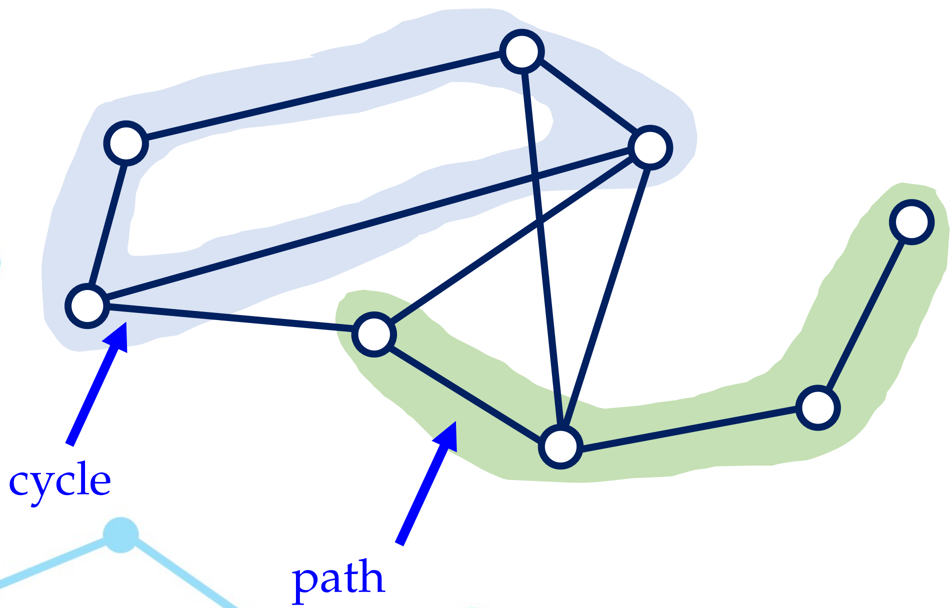
Non-simple graph



Simple graph

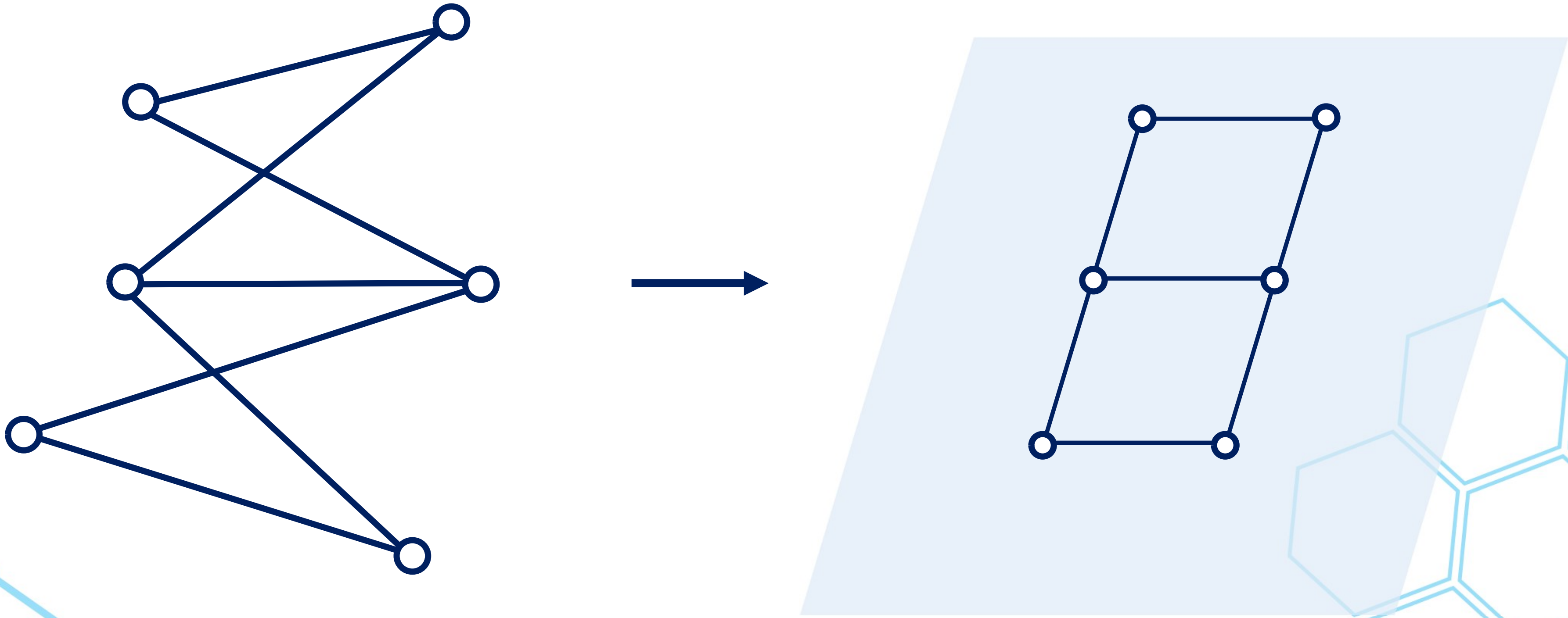
More Graph Terminology

Cycles and paths are alternating sequences of vertices and edges which are exactly what you expect. Only one note: we don't allow repeated vertices in paths and cycles, those that allow repeated vertices are called walks and circuit respectively (we won't use those).

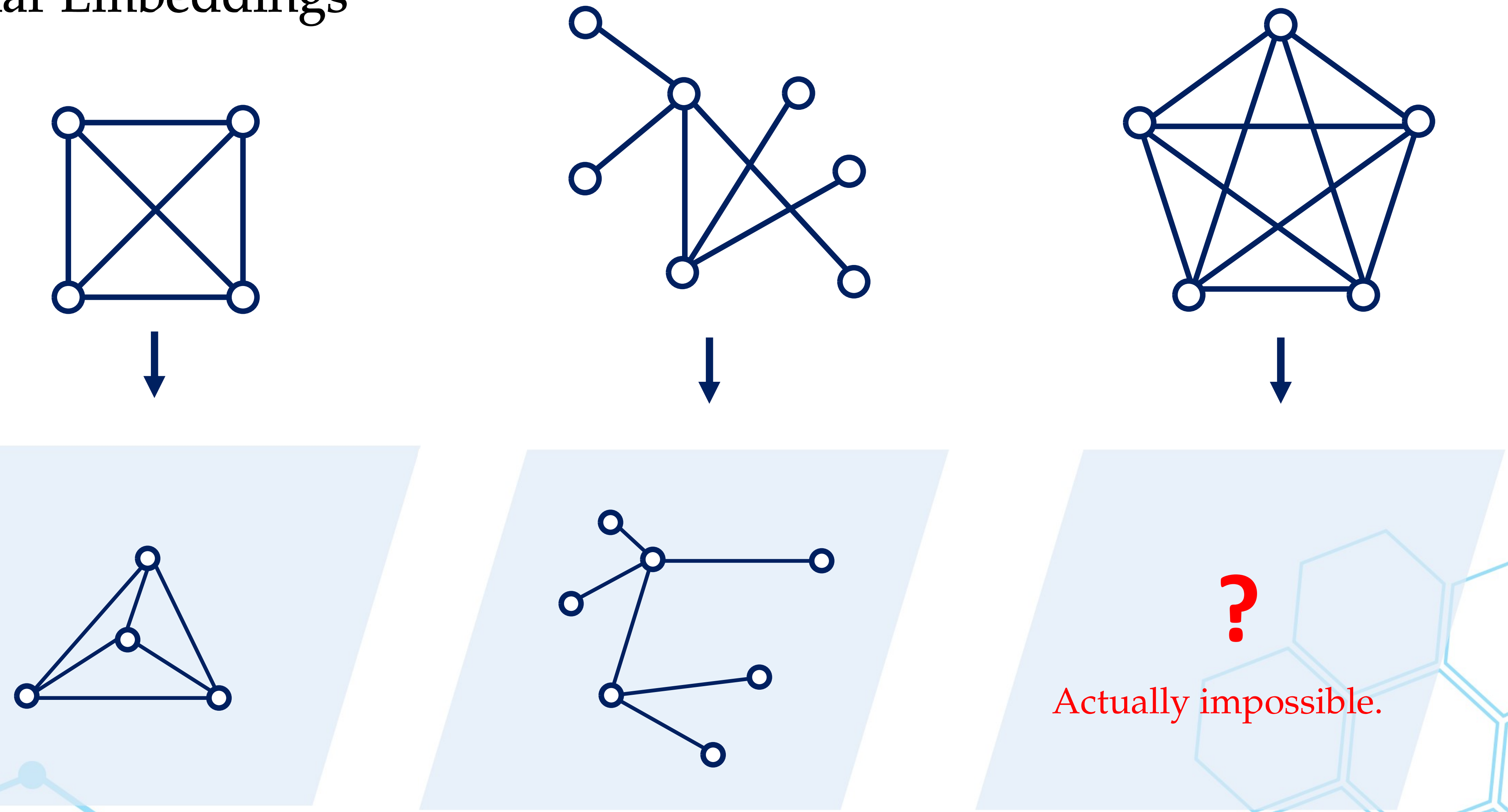


Planar Embeddings

A planar embedding of a graph G is a drawing of G on the plane so that vertices become points, edges become curves and no two such curves intersect except at the vertex points.



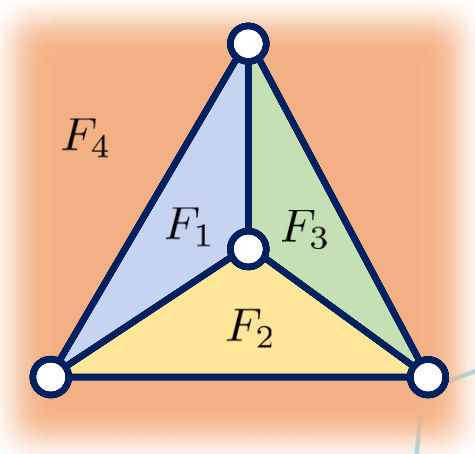
Planar Embeddings



Planar Embeddings

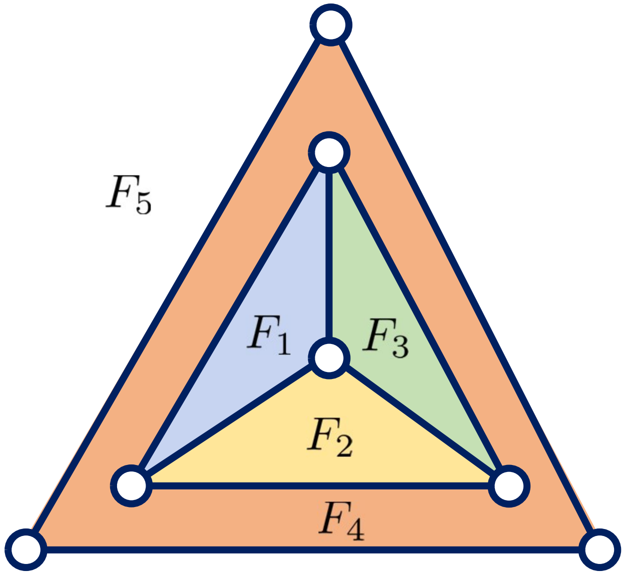
Throughout this lecture, think of my screen as a plane. All my graph drawings will be planar embeddings unless otherwise stated.

In a planar drawing, edges (and vertices) cut the plane into several different regions. These regions are called faces.



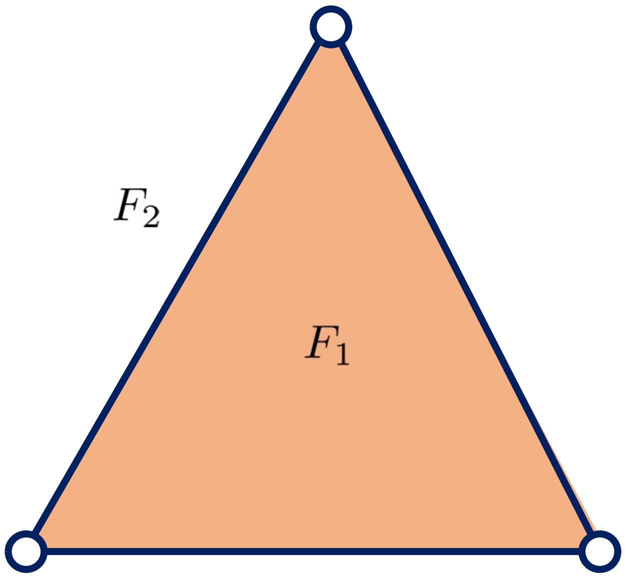
So, we can also talk about $\#V - \#E + \#F$ for graphs that has a planar embedding (technical term for those graphs is 'planar graphs').

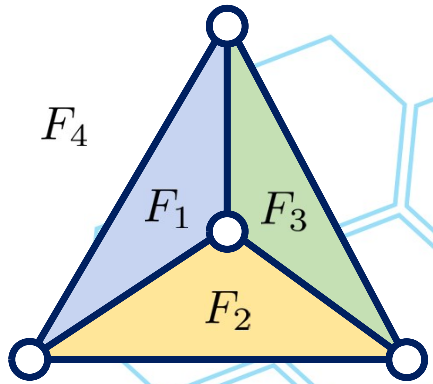
It's not always 2



In this disconnected graph, $\#V - \#E + \#F = 1$.

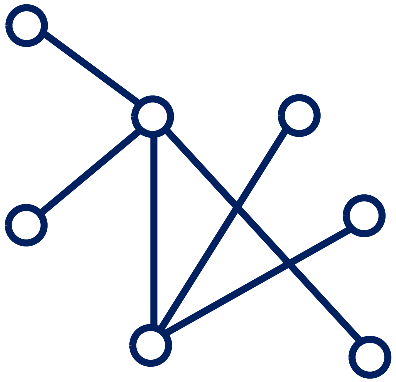
But, if you "break" it into components, it seems like the formula still works for EACH component...



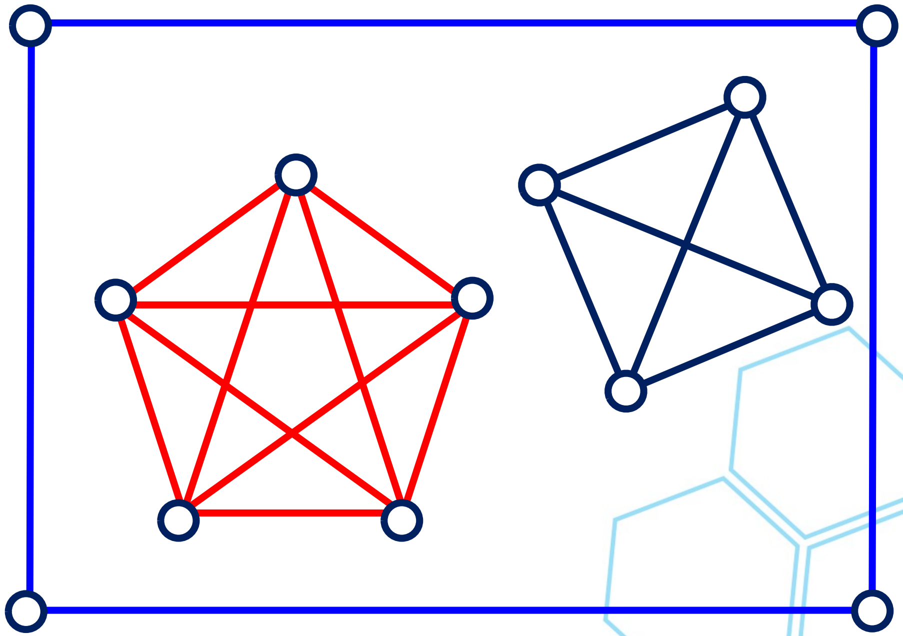


More terms…

A graph (planar or not) is called connected if any two vertices are joined by some path. Otherwise, the graph is called disconnected. The maximal connected subgraphs are called connected components.



A connected graph



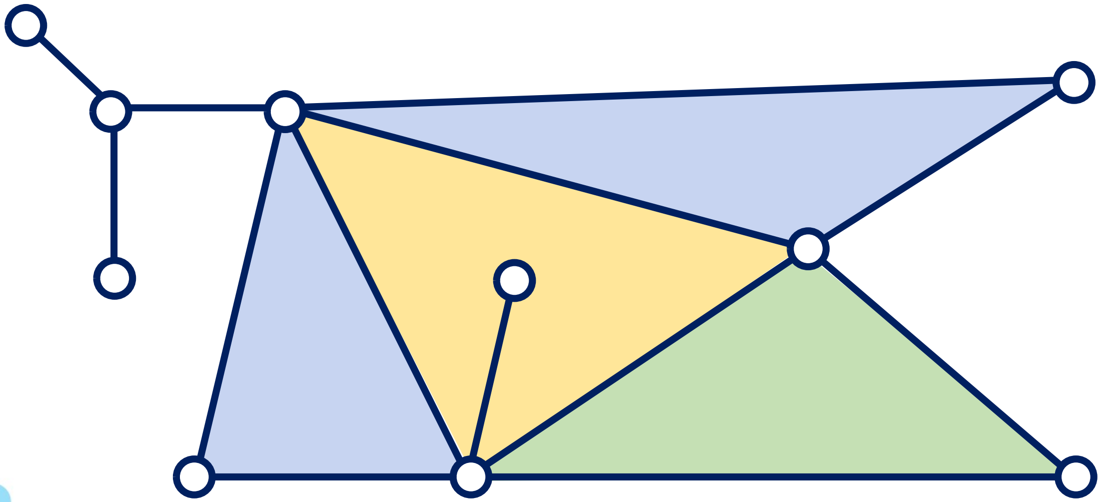
A disconnected graph and its components

The Main Theorem

Theorem. Let G be any connected planar graph. Then, its planar embedding satisfies

$$\#V - \#E + \#F = 2$$

where $\#V$, $\#E$ and $\#F$ are number of vertices, edges and faces respectively. In particular, any two planar embedding of the same graph have the same number of faces.

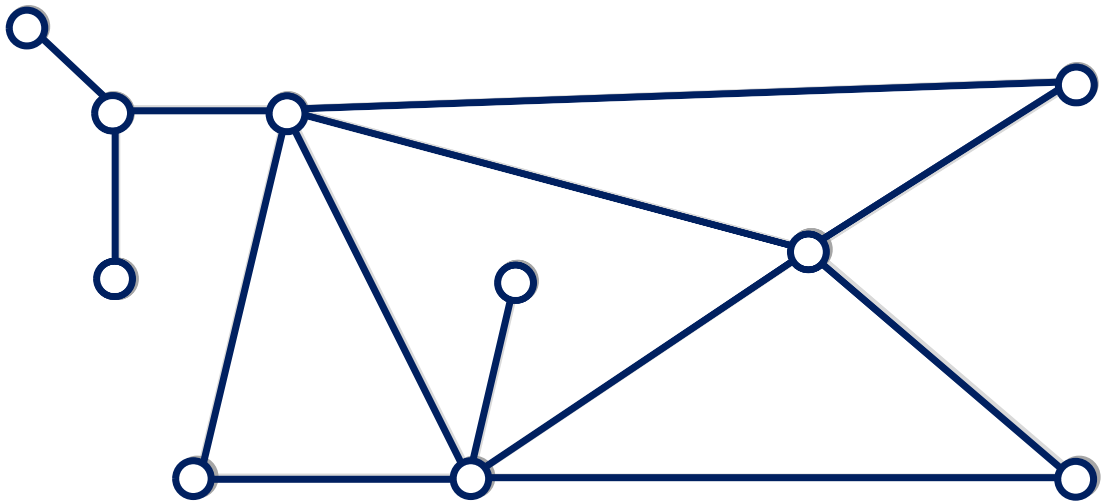


Intuitive Proof Sketch

- We grey out the planar embedding just like in a drawing book, and then trace out step by step.

- Start by redrawing one of the vertices. At this stage, $\#V - \#E + \#F = 1 - 0 + 1 = 2$.

- Redraw the edges (together with the vertices on its ends if necessary) one by one while keeping your "new drawing" connected.

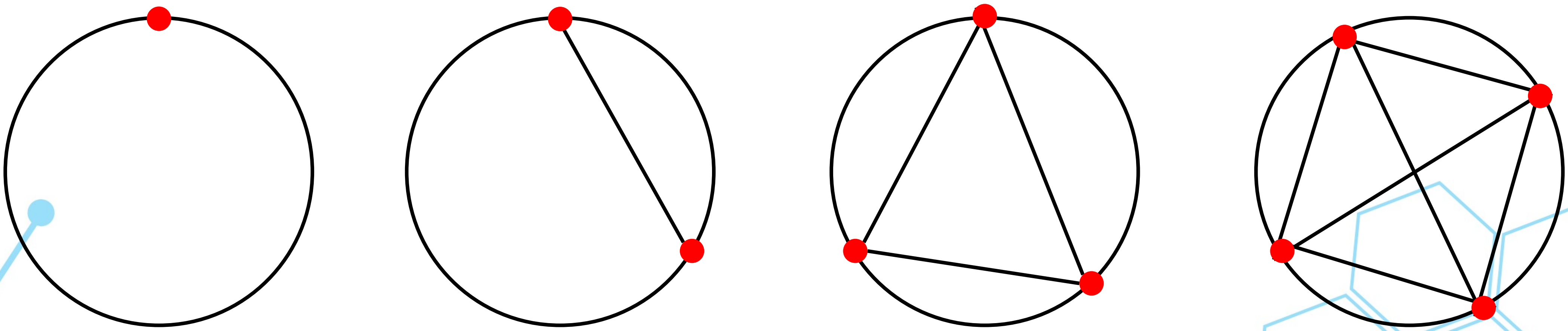


- After each step, you either add one edge and one vertex, or add one edge and one face. Hence, the quantity $\#V - \#E + \#F$ does not change.

- Since the graph is connected, your drawing will eventually be complete. Hence, $\#V - \#E + \#F = 2$ in the end, too!

That Tricky Pattern

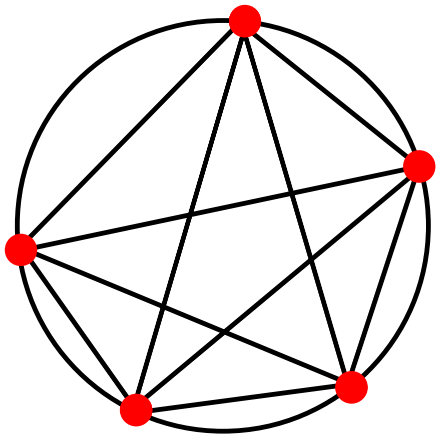
Let there be n points on the circle so that no three chords joining these points are concurrent. All the chords are then drawn, dividing the interior of the circle into multiple regions. How many regions are there?



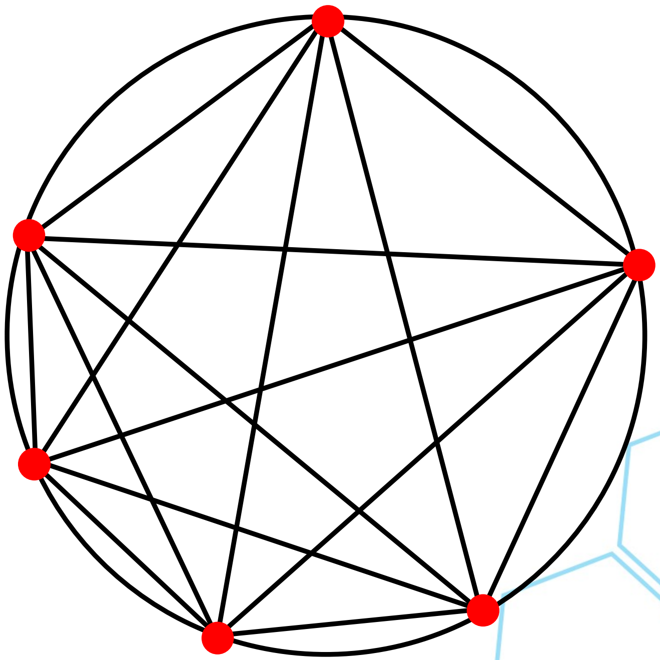
1 region 2 regions 4 regions 8 regions

That Tricky Pattern

We have seen that the answer is not 2^n. The pattern breaks at $n = 6$.



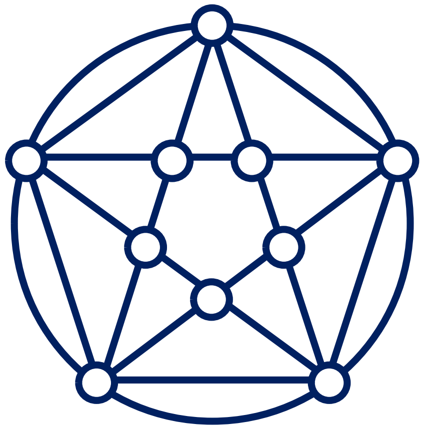
16 regions



31 regions

Computing Regions is Difficult

We can think of the picture as a planar embedding of some graph.

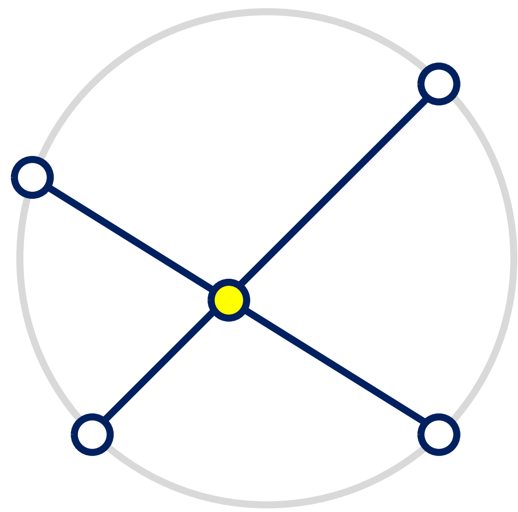


Computing regions is very difficult. But, by $\#V - \#E + \#F = 2$, it suffices to compute $\#V$ and $\#E$.

Computing $\#V$

- There are n vertices on the circle.

- Each interior vertex is formed by intersection of two chords.



- Whenever we select 4 points on the circle, there is exactly one pair of chords between them intersect in the interior.

- Hence, there are $\binom{n}{4}$ vertices in the interior.

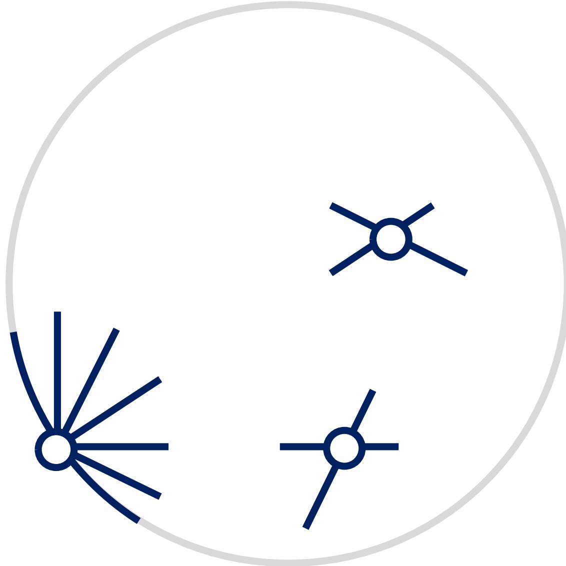
$$\boxed{\#V = n + \binom{n}{4}}$$

Computing $\#E$

- Each vertex on the circle shoot out $n+1$ pieces of $\#E$.

- Each vertex in the interior shoot out 4 pieces of $\#E$.

- So, in total, $\#E = (n+1) \times n + 4 \times \binom{n}{4}$???



No, each edge is counted twice!

$$\boxed{\#E = \frac{(n+1)n + 4\binom{n}{4}}{2}}$$

Final Answer

Now, just substitute into $\#V - \#E + \#F = 2$.

$$n + \binom{n}{4} - \frac{(n+1)n + 4\binom{n}{4}}{2} + \#F = 2 \longrightarrow \#F = \binom{n}{4} + \binom{n}{2} + 2$$

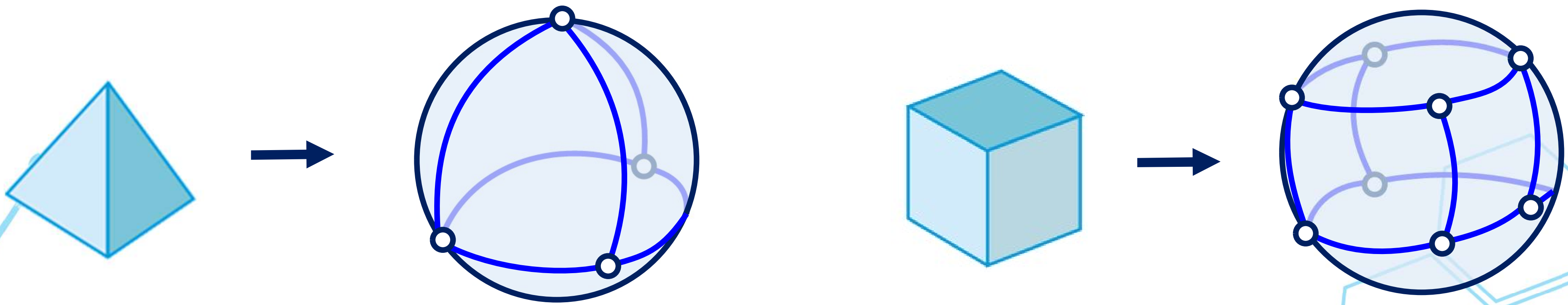
Remember, $\#F$ includes the outer face. Hence, the answer is…

$$\boxed{\binom{n}{4} + \binom{n}{2} + 1}$$

Connection with Polyhedra

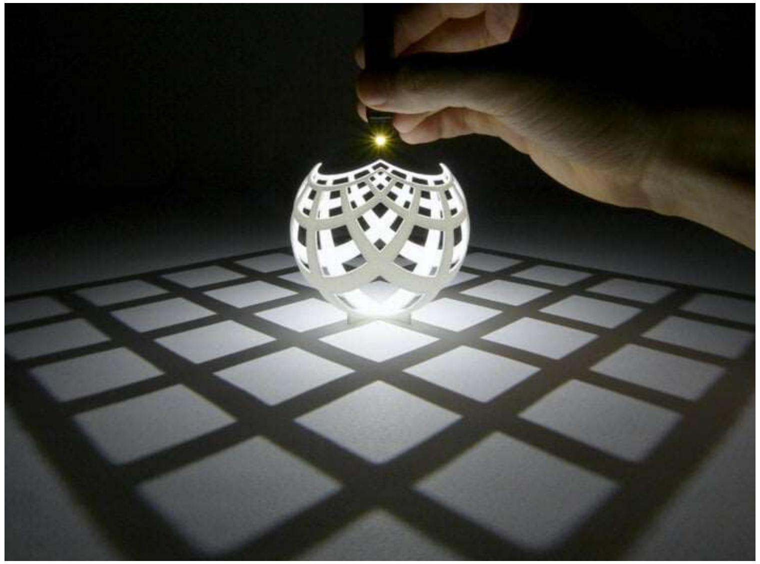
Think of the polyhedra as if their surfaces are made-up with paper.

Then, we can blow air inside them (or if you prefer, expand the air inside them). This results in following pictures:



Connection with Polyhedra

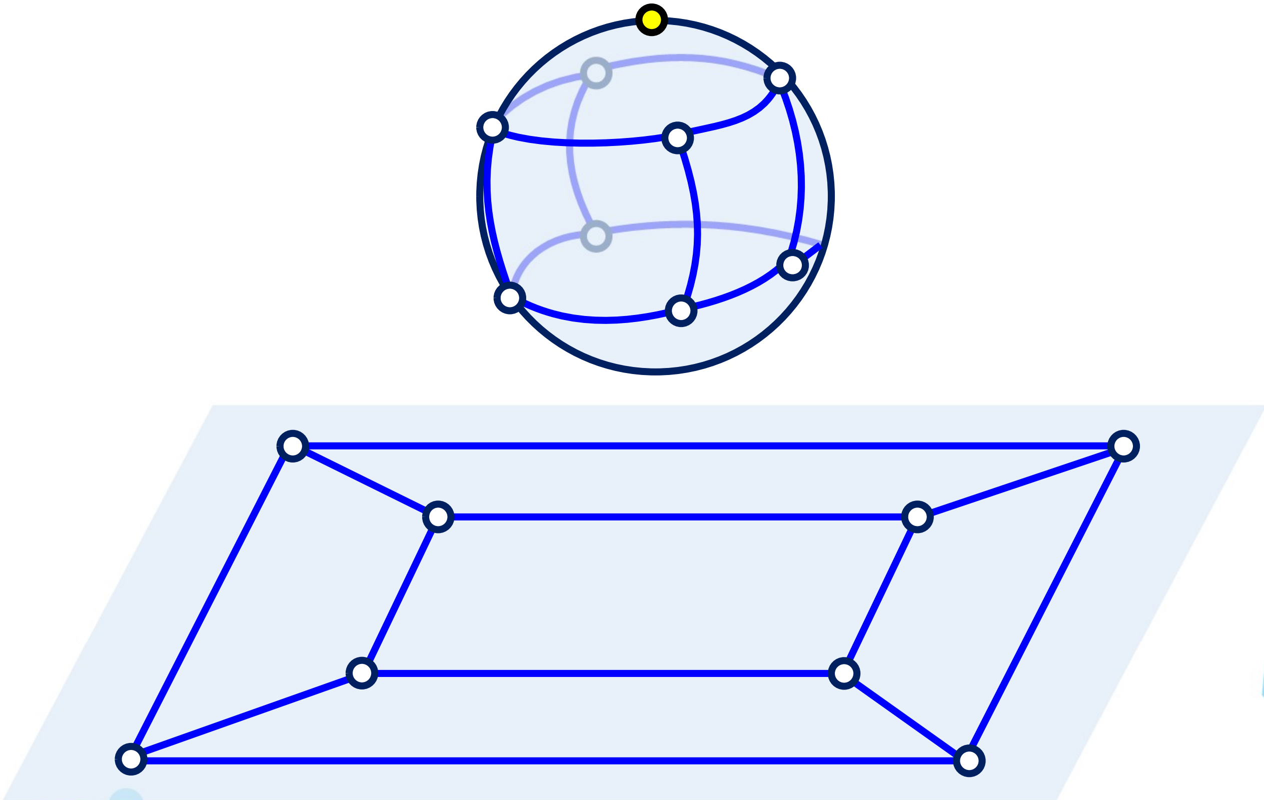
Now, reorient the sphere so that the north pole lies strictly inside some face. Put the sphere (made with transparent material) above the plane somewhere, then shine a light at the north pole.



Something like this….

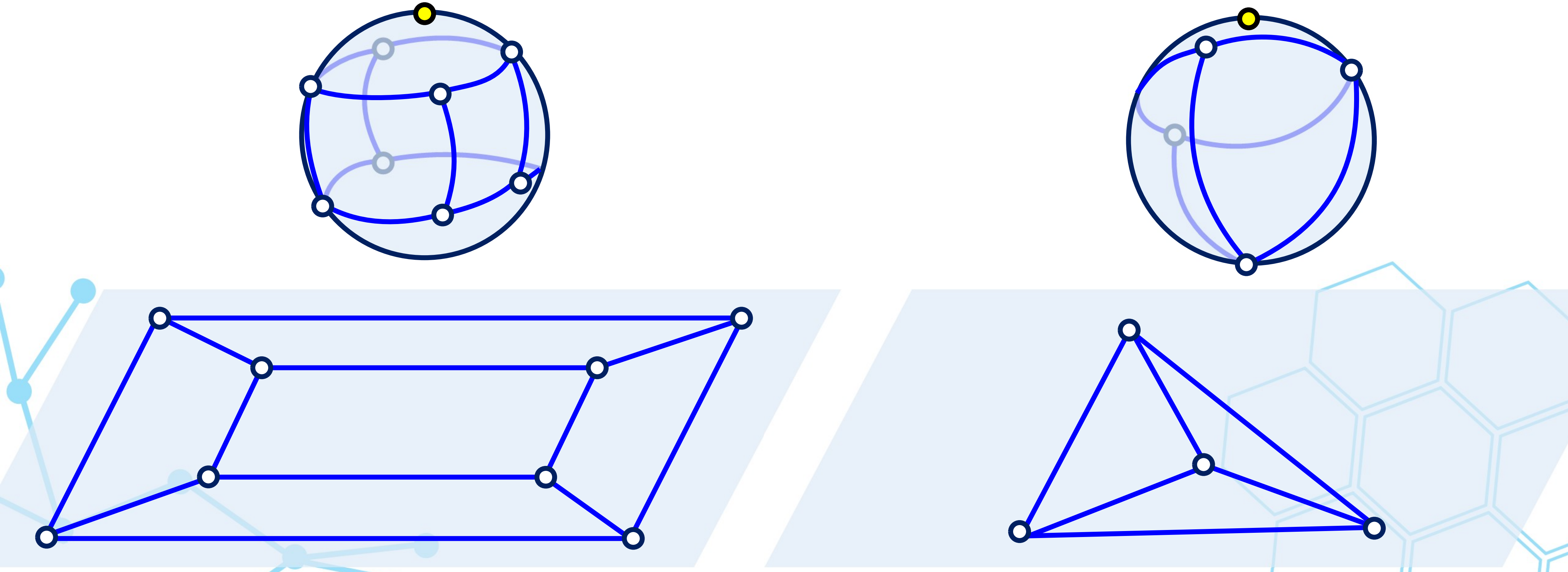
Connection with Polyhedra

Then, the shadow of the vertices and edges form a planar embedding of some graph.



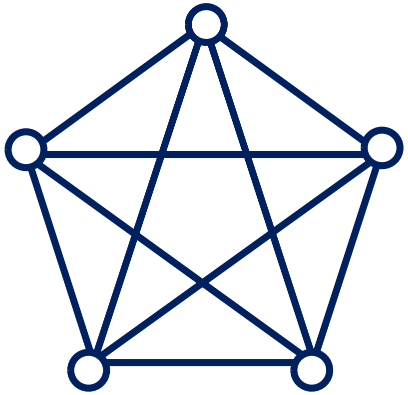
Connection with Polyhedra

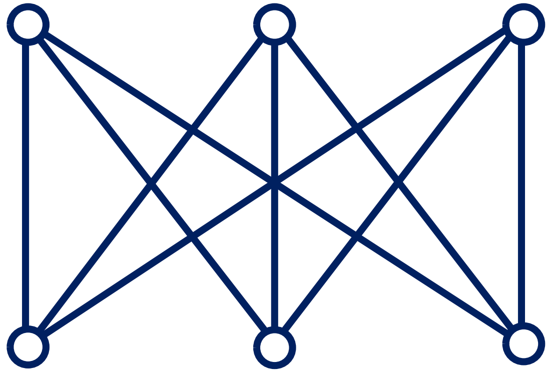
There is a bijection between vertices, edges and faces on the blown-up polyhedra and vertices, edges and faces of the shadow planar graph. So, $\#V - \#E + \#F = 2$ on the polyhedra, too!



Bonus: Impossibility of an Embedding

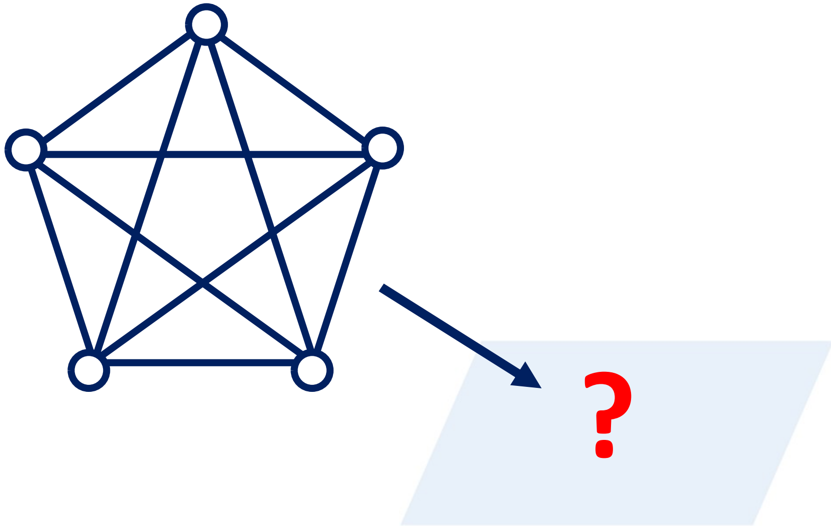
We have claimed that it is impossible to embed the following graphs into the plane.



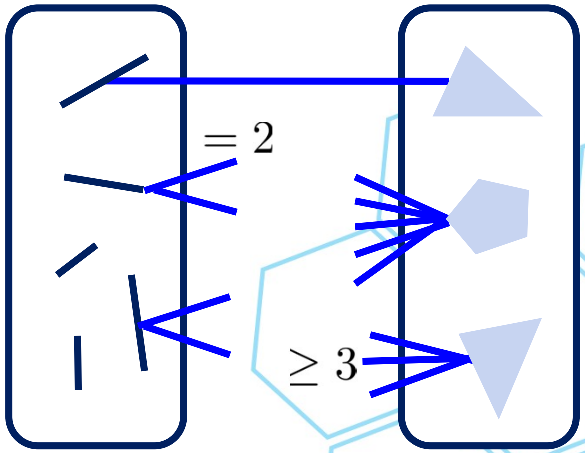


We can prove this using $\#V - \#E + \#F = 2$. I'll do it for the graph on the left. The one on the right is exercise if you want.

Solution: Just Double Count



- Suppose that we can embed it into the plane somehow.
- Count the pairs (e, f) of an edge e and a face f so that e is a boundary segment of f.
- This gives us $2\#E \geq 3\#F$.
- But, this inequality can be transformed into an inequality between $\#E$ and $\#V$ by using $\#V - \#E + \#F = 2$.



$$2\#E \geq 3(2 + \#E - \#V) \longrightarrow 2 \cdot 10 \geq 3(2 + 10 - 5)$$

Impossible!

That's it for today.

See you tomorrow!