We will begin at 1:05 MMT

Read this problem while we wait…

A soccer ball has 12 pentagons and 20 hexagons. Using this information and the picture given to the right, can you find the number of vertices and edges?

Record the meeting…

Some Housekeeping

- I have heard that many of you are experiencing your final exams or IG exams at the moment. So, I decided to extend the deadlines as follows:
  - Deadline of H_2 is now beginning of P_4 ,
  - Deadline of H_3 is now beginning of P_5 ,
  - H_4 is now optional.

- I will allow those who satisfied the quota of H_2 early to change the diamond-quota of their preferred problem set into 1.

Content so far…

I
- L1: Monovariants
- L2: Invariants
- L3: Alternating-variants

II
- L4: Inductive constructions
- L5: Greedy and RUST

III
- L6: Counting in two ways
- L7: Inequalities and bounding
- L8: Counting in graphs
- L9: Injections and bijections

IV
- L10: Pigeonhole principle
- L11: Continuity and descent
- L12: Leveraging symmetry

V
- L13: Combinatorial games

VI
- L14: Combinatorial geometry

VII
- L15: Results in graph theory I
- L16: Results in graph theory II

Lecture – 6

Counting in Two Ways

The First Example

Let's try to show, without using algebra, that $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

(Un)related Question: Suppose there are $n + 1$ people playing a round-robin tennis tournament. How many matches are played?

- One person plays n games, another person keeps playing $n - 1$ games, etc.

$$\text{Number of matches} = 1 + 2 + 3 + \cdots + n.$$

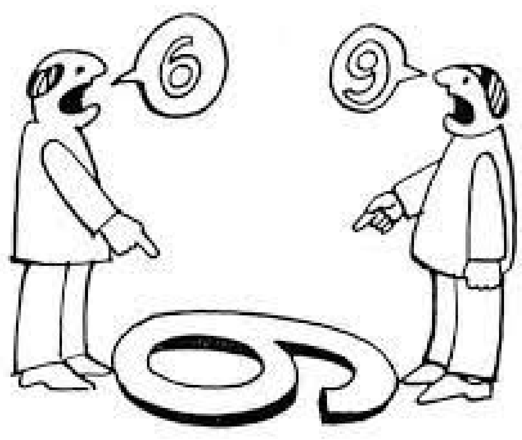
- Everyone plays n games. Every game is played by 2 people. Therefore,

$$\text{Number of matches} = \frac{n(n+1)}{2}.$$

← A.k.a. $\binom{n+1}{2}$

So what is double counting?

You literally count the same object in two different perspectives. This gives us a relation between the perspectives.



Skilled Students Take the IMC (IMC 2002)

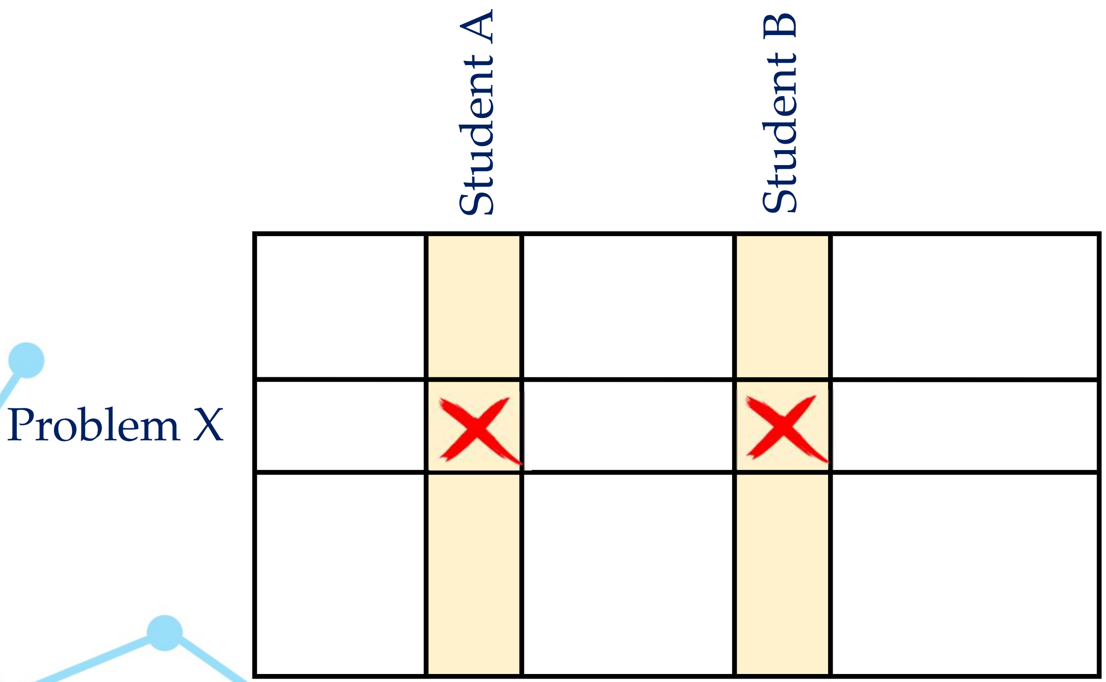
Two hundred students participated in a mathematical contest. They had six problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these students.

Problem 1	✓	✗	✗	✗	✓	✗	✓	✗	✓	✗
Problem 2	✓	✗	✗	✗	✗	✗	✓	✗	✓	✗
Problem 3	✗	✓	✗	✗	✗	✗	✗	✗	✗	✗
Problem 4	✗	✗	✓	✗	✗	✓	✗	✗	✓	✗
Problem 5	✗	✓	✗	✗	✗	✓	✗	✗	✗	✗
Problem 6	✗	✓	✗	✗	✗	✗	✓	✗	✓	✗

Constructing Correspondence

Let's say that conclusion is false.

Then, for any two students, there is a problem that they both didn't solve.



Intuition

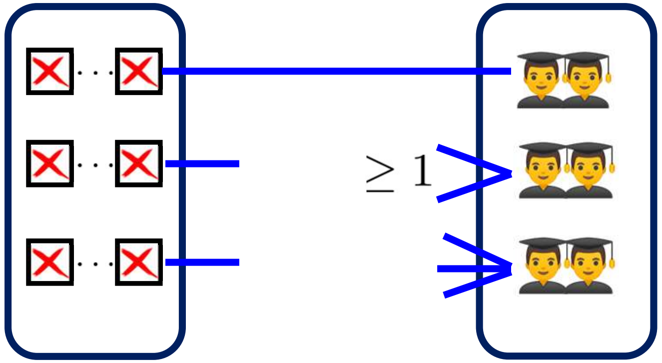
- Every student solved ≥ 120 problems. So, there can't be too many  .
- Every two student meet at  . So, there can't be too little  .

Constructing Correspondence

Count the number of pairs of the form  .

Columns perspective:

$$\text{Number of pairs} \geq \binom{200}{2}.$$

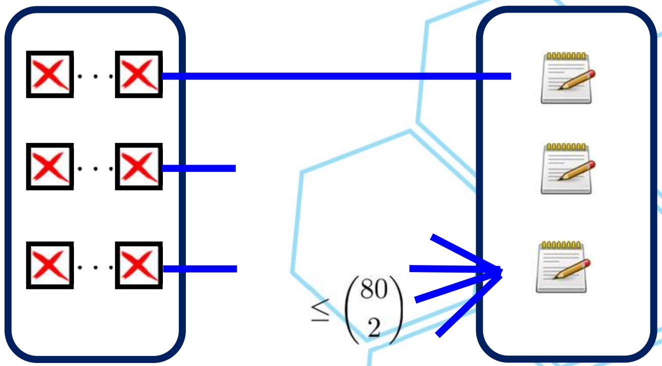


Rows perspective:

$$\text{Number of pairs} \leq 6 \times \binom{200-120}{2}.$$

$$\binom{200}{2} \leq 6 \times \binom{80}{2} \longrightarrow 19900 \leq 18960$$

Impossible!

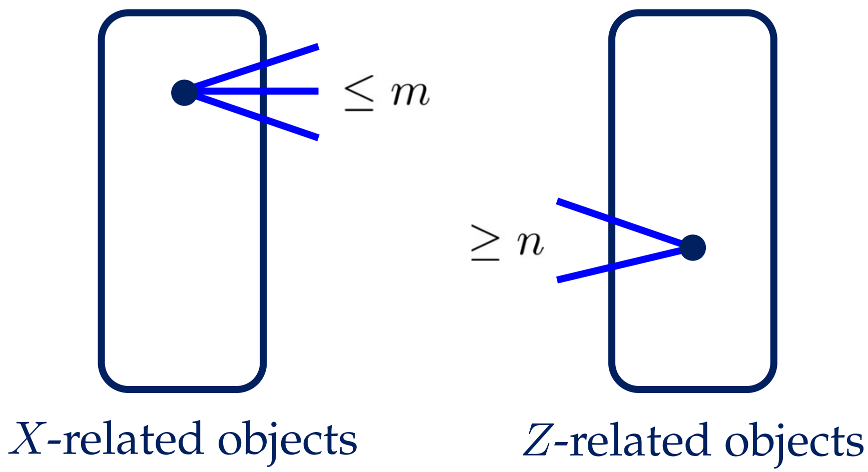


Usual Template of Counting Arguments

We want to relate two quantities X and Y. Then, we find the quantity Z and establish relationships between X, Z and Y, Z.

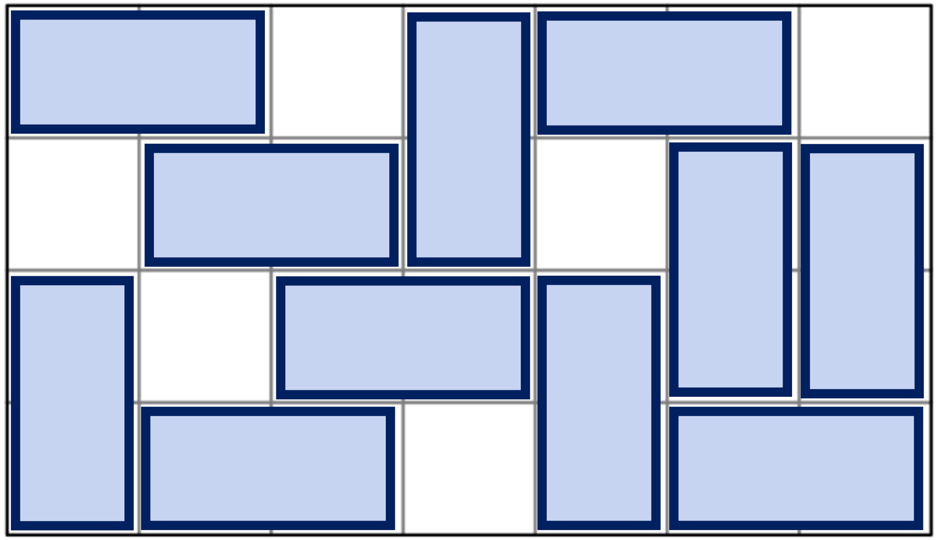
The way we relate two quantities (say X and Z) usually look like this:

- Establish a correspondence (relation) between X-related things and Z-related things.
- Every (X-related thing) corresponds to (at least/most) m (Z-related things).
- Every (Z-related thing) there are (at most/least) n (X-related things).



Singapore Open MO 2019/P5

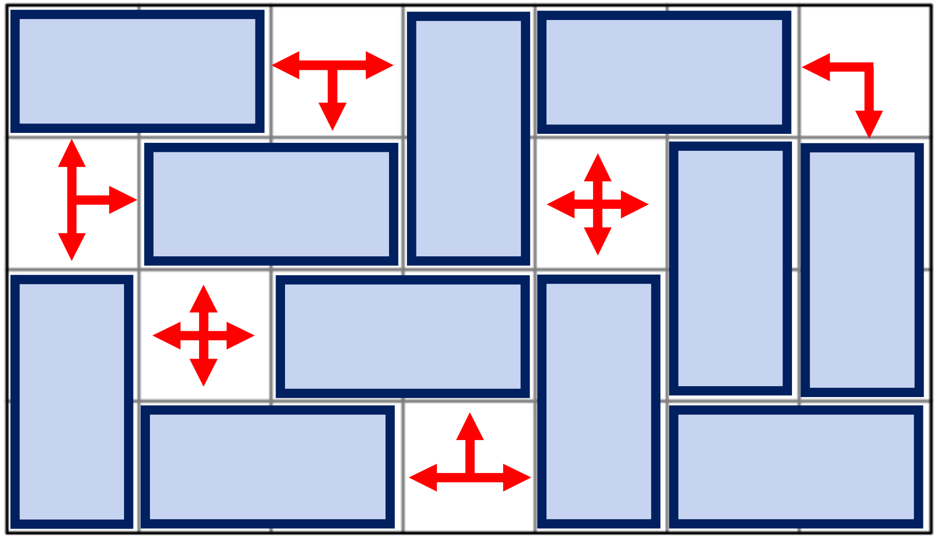
In a $m \times n$ chessboard with $m, n \geq 3$, some dominoes are placed (without overlap) with each domino covering exactly two adjacent cells. Show that if no more dominoes can be added to the grid, then at least $2/3$ of the chessboard is covered by dominoes.



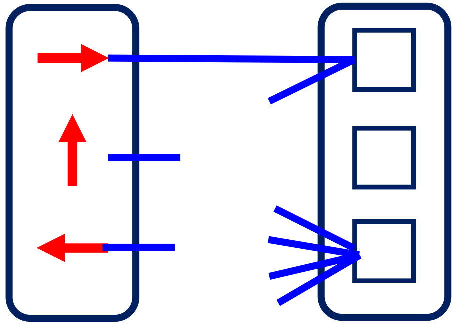
Constructing correspondence

Let v be the number of vacant squares. Let's cook up some relationship between v and $mn - v$.

Idea: For each vacant square, shoot 2, 3, 4 lasers in cardinal directions depending on its position: at corner, at the edge or in the interior.



Constructing correspondence

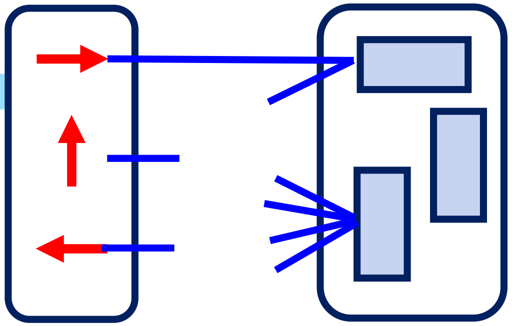


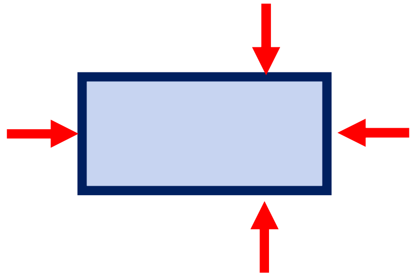
Let's count the number of lasers.

Vacant square perspective: Each vacant square shoots at least 2 lasers. So,

Number of lasers $\geq 2v$.

Domino perspective: Each domino receives at most 4 lasers (≤ 1 from each side).





Not good enough

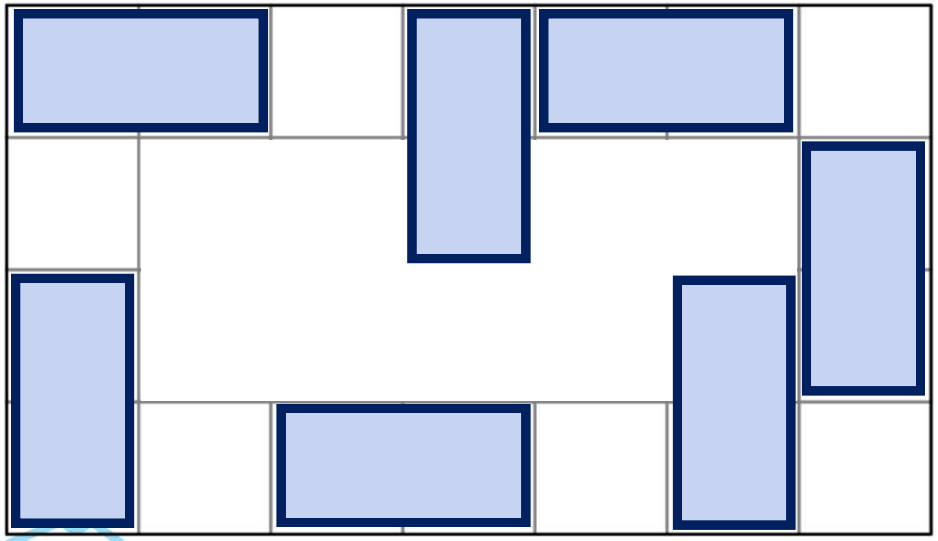
$$v \leq \frac{mn}{2}$$

Number of lasers $\leq 4 \times$ number of dominoes $= 2(mn - v)$.

Improvement

Improvement: If we can upper bound the number of dominoes and vacant squares on the boundary, we can improve our result.

Let there be X vacant squares and Y dominoes intersecting the boundary. Let there be c vacant corners.



This gives us the following.

Vacant Square Perspective:

$$\text{number of lasers} = 4v - X - c$$

Domino Perspective:

$$\text{number of lasers} \leq 4\frac{mn - v}{2} - Y$$

Improvement

Upon combining, we have

$$4v - X - c \leq 2mn - 2v - Y \longrightarrow v \leq \frac{mn}{3} + \frac{X-Y}{6} + \frac{c}{6}$$

Question: How big are $X - Y$ and c ?

Answer

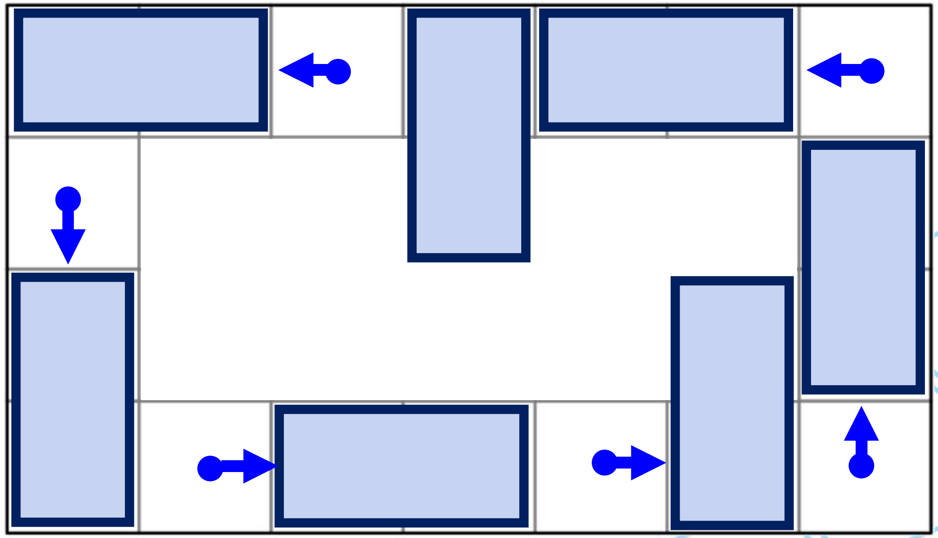
Obvious: $c \leq 4$.

Less obvious: $X - Y \leq 0$.

Almost!

$$v \leq \frac{mn}{3} + \frac{2}{3}$$

Walk along the border clockwise. Every vacant square is followed by a domino.



Resolving the Annoyance

What we have: $v \leq \frac{mn}{3} + \frac{2}{3}$ ⟷ $3v \leq mn + 2$

What we need: $v \leq \frac{mn}{3}$

So, we just need to consider the cases $3v = mn + 2$ and $3v = mn + 1$.

↑ Impossible because $mn - v$ is even.

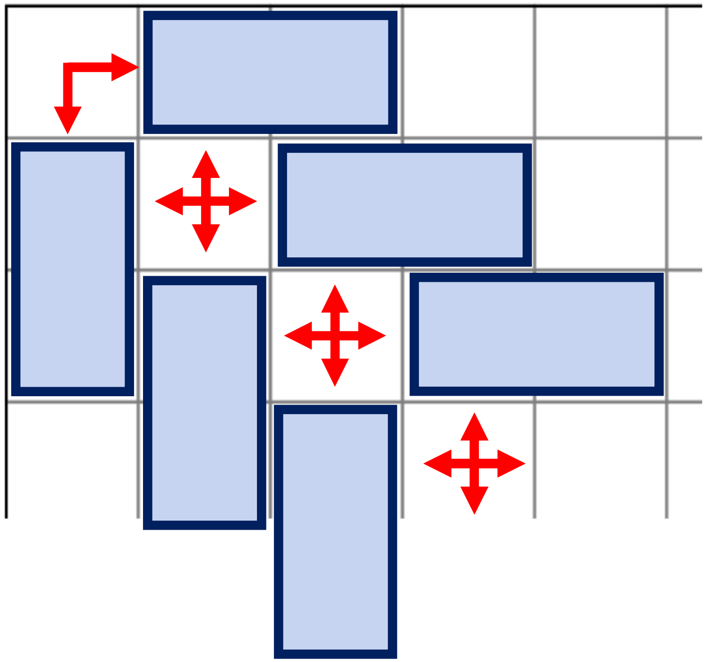
So, suppose that $3v = mn + 2$. Then, we have equality case in ALL of our inequalities.

- $c = 4$ → Every corner is vacant.
- $X = Y$ → Vacant squares and dominoes alternate on the boundary
- Every domino in the interior gets hit by four lasers.

TIP: (Un)expected equality cases in inequality usually gives us huge amount of information.

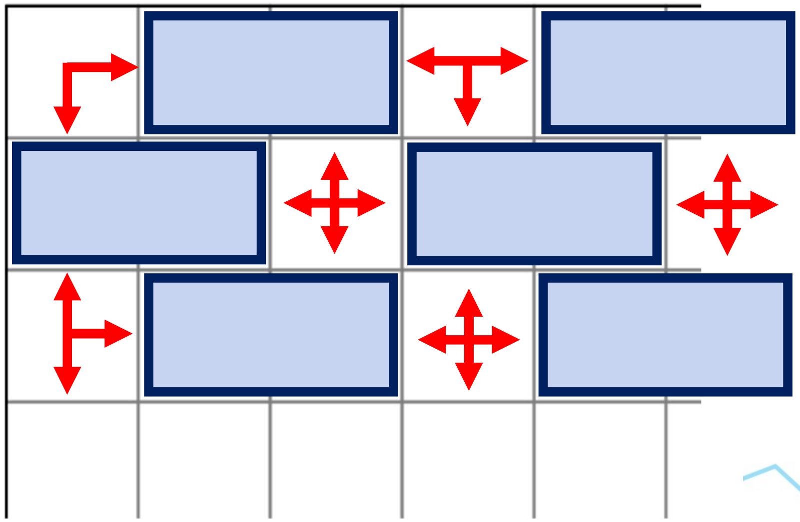
Resolving the Annoyance

Now, we can pin down the tiling using these information.



Case 1

In this case, the tiles will venture off to infinity. So, impossible.



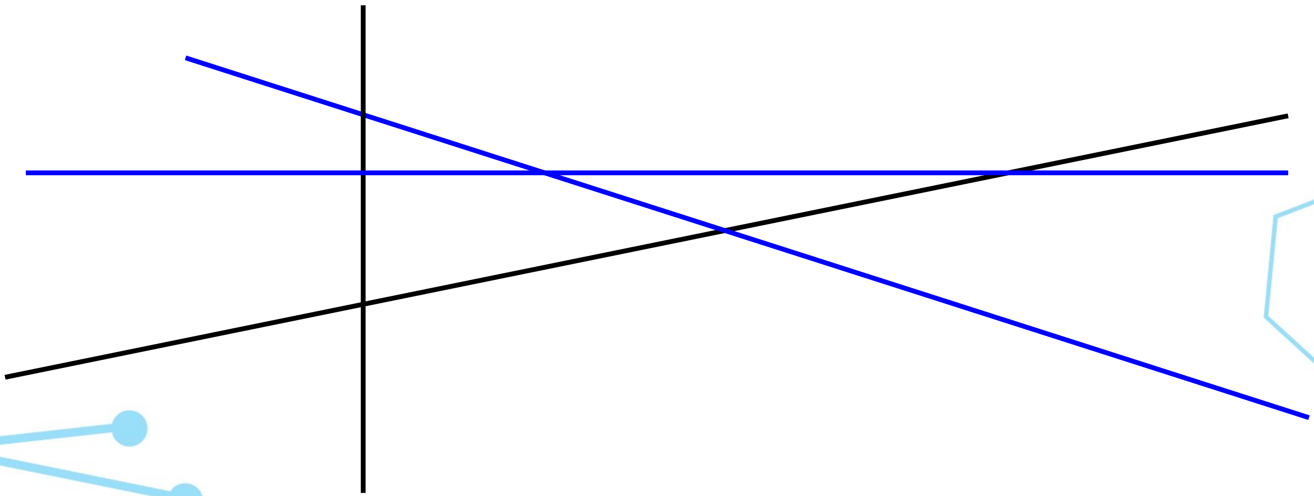
Case 2

In this case, top right corner will not be vacant. So, impossible.

Bule Lines (IMO 2014/P6)

A set of lines in the plane is in *general position* if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its *finite regions*. Prove that for all sufficiently large n, in any set of n lines in general position, it is possible to colour at least $\sqrt{n}$ lines blue in such a way that none of its finite regions have completely blue boundary.

Note: Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant c.



Where to start…

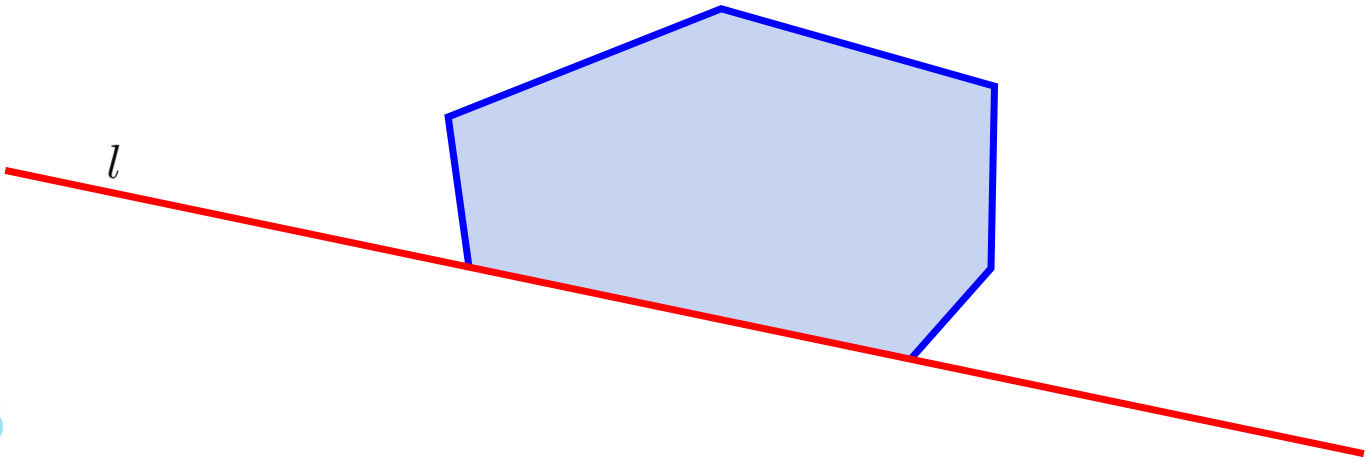
Starting position can be very arbitrary. It is really difficult to know what lines to choose.

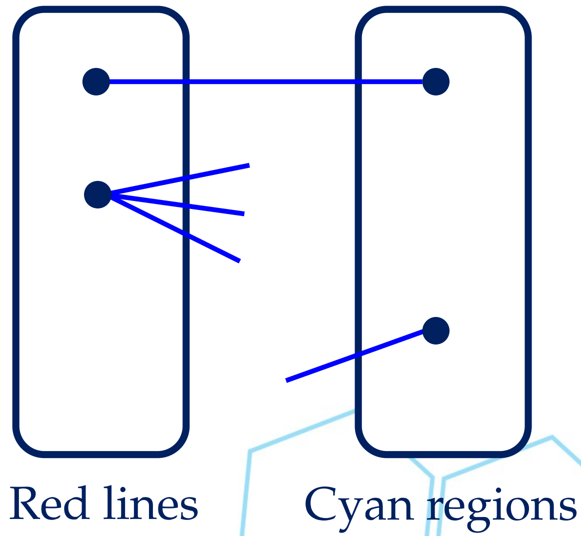
Greedy Choice: Look at the lines one by one, colour it blue if it doesn't create conflict.

This gives us many(?) lines, but how many?

Setting the correspondence

Consequence of Greedy: Every red line l touches a region where l is the only red boundary edge.





Call these kind of regions cyan. Then, we get the lower bound perspective:

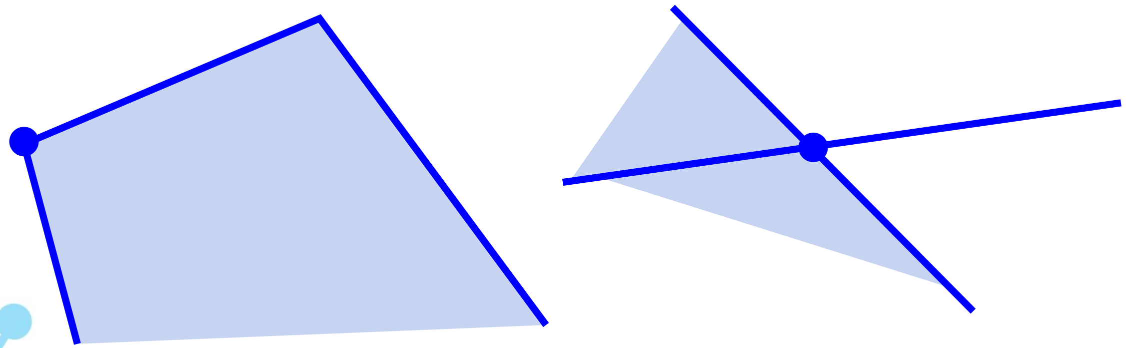
$$\text{Number of cyan regions } \geq n - k \quad .$$

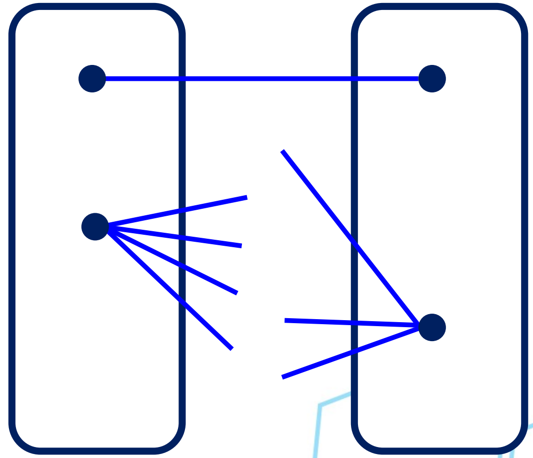
Setting the correspondence

IMPORTANT: blue intersections are easy to count.

Upper bound perspective:

- Every cyan regions contain a blue intersection.
- But, every blue intersection is included in at most 4 cyan regions.





Blue points Cyan regions

This gives us this:

$$\text{Number of cyan regions } \leq 4 \times \binom{k}{2} \ .$$

Combining Results

Upon combining, we get

This term is unimportant i.e. yeet-able.

$$n - k \leq 4\binom{k}{2} \longrightarrow n \leq 2k^2 - 2k \longrightarrow k \geq \frac{1}{\sqrt{2}} \cdot \sqrt{n}$$

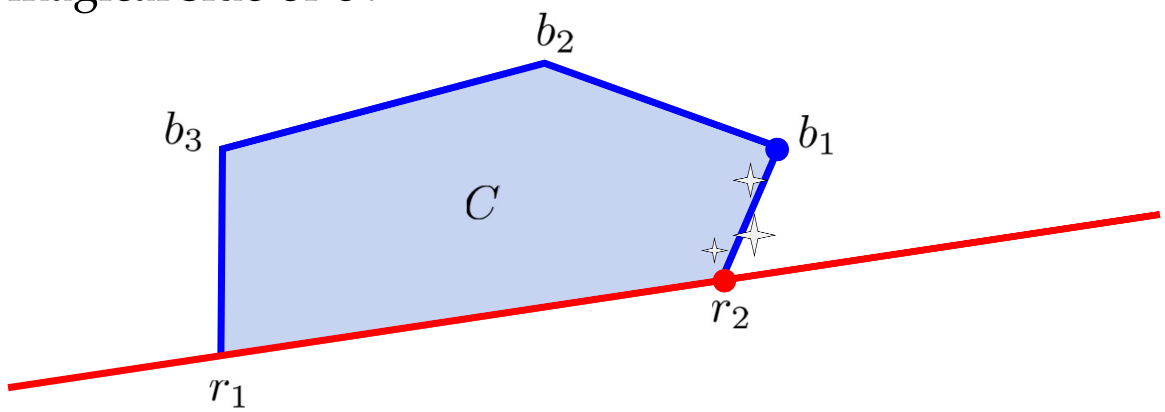
Of course, an IMO6 wouldn't die this easily!

Note: The result that we get i.e. $c = 1/\sqrt{2}$ is the original proposed question. The problem selection committee managed to improve the result to $c = 1$. The solution for $c = 1/\sqrt{2}$ is worth 3 points and the solution for $1/\sqrt{2} < c < 1$ is worth 4 points.

Improvement

Call a point red if it is an intersection of a blue and a red line.

For each cyan region C, let its vertices be $r_1, r_2, b_1, b_2, \ldots, b_k$ in counter-clockwise order. We shall call the side $r_2 b_1$ as the magical side of C.



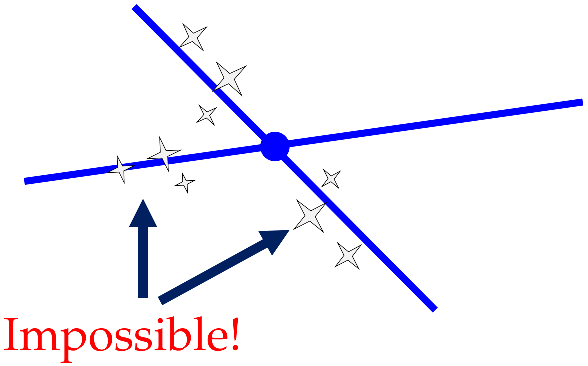
Note: There are same number of cyan regions as there are magical sides. So,

$$\text{Number of magical sides } \geq n - k \; .$$

Improvement

Now, counting from blue points gives a better bound.

Main Observation: Each blue point corresponds to at most 2 magical sides.



Therefore, we have this:

$$\text{Number of magical sides} \leq 2 \times \binom{k}{2}.$$

Improvement

Upon combining, we have

$$n - k \leq \text{ Number of magical sides } \leq 2 \times \binom{k}{2}$$

Therefore, $n - k \leq k^2 - k$ and hence $k \geq \sqrt{n}$.

Remark: In fact, we can improve c to be infinity, but $c = 1$ is the best if we use greedy. Using nukes from Extremal Graph Theory, we can get $\sim C\sqrt{n \log n}$ blue lines.

↑

Whether or not this is the best
is still an open problem.

It's not about the length

IMO2014/P6 solution is so short that you can probably write it down on your hand. And it only uses very basic of basic math!

> We claim that a greedy algorithm works. Start with all red lines, and color lines blue until coloring any other line blue causes an all blue region. Suppose b of the lines were colored blue.
>
> Let ℓ be a red line. Coloring it blue would make some region completely blue, so ℓ bounds a region that has all but one of its edges blue. Call this region P. Note that all regions are convex polygons, and let v be a vertex that is one away from the edge that is on ℓ. Arbitrarily assign exactly one such v to each ℓ, and call it an anchor of ℓ. It is not hard to see that an anchor must be at the intersection of two blue lines, and that for each blue/blue intersection, it is the anchor of at most two red lines. Thus,
>
> $$2\binom{b}{2} \geq n - b \implies b \geq \sqrt{n},$$
>
> as desired.

But that doesn't mean the problem is easy. Just look at the statistics.

	P1	P2	P3	P4	P5	P6
Num(P# = 0)	75	240	479	24	301	514
Num(P# = 1)	23	32	43	103	60	7
Num(P# = 2)	14	25	1	28	83	7
Num(P# = 3)	22	17	2	16	10	11
Num(P# = 4)	15	14	3	5	8	0
Num(P# = 5)	18	39	0	3	3	5
Num(P# = 6)	23	71	4	3	11	1
Num(P# = 7)	370	122	28	378	84	15

Only these people solved the problem.

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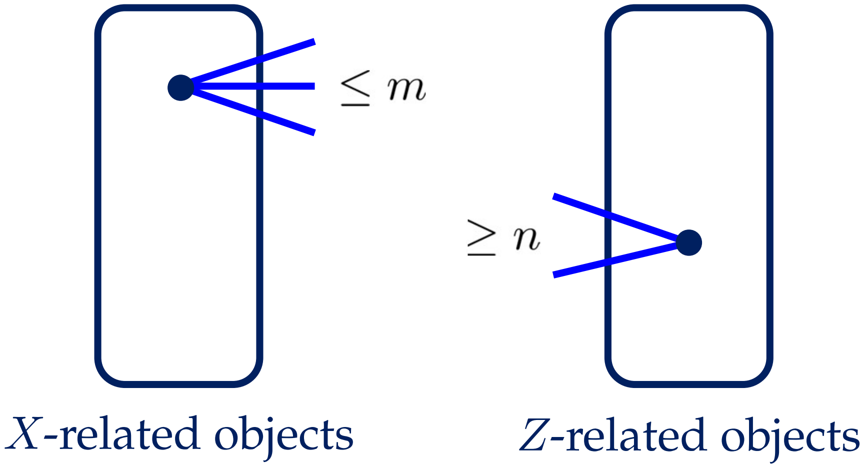

That's it for this lecture.

See you on Problem Solving Session!