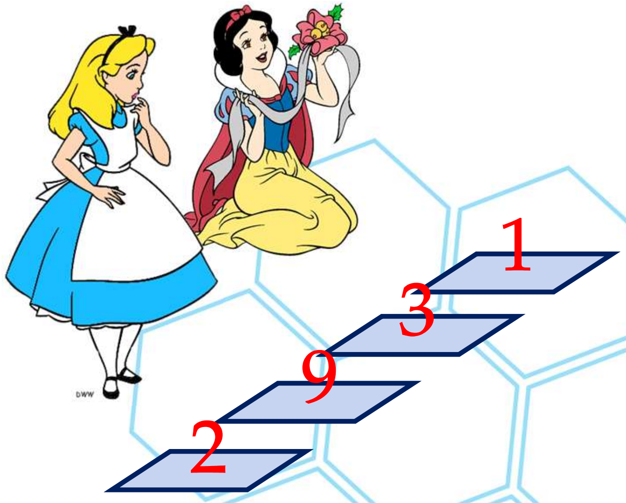
We will begin at 8:35

While we wait, try this problem :)

Alice and Snow play a game with $2n$ cards, on each of which is written a positive integer. The pack is shuffled and the cards laid out in a row, with the numbers facing upwards. Alice starts, and the girls take turns to remove one card form either end of the row, until Snow picks up the final card. Each girl's score is the sum of the numbers on her chosen cards at the end of the game, and the girl with higher score will win (if scores are equal, there is no winner). Prove that Alice can design a strategy so that Snow never wins.



Some housekeeping

- Problem Set 1 diamond-problems are due tomorrow (you can submit heart-problems at any time).

- Starting from Problem Set 2, the diamond quota will be mandatory for IMO team members.

- I changed the topics of lectures 11 and 12.

- Tomorrow's lecture and Problem Solving Session will make extensive use of mathematical induction. Today is the last chance to study if you haven't. Resources are posted as announcements on Google Classroom.

Lecture – 3

Alternating-variants

Magic Trick…

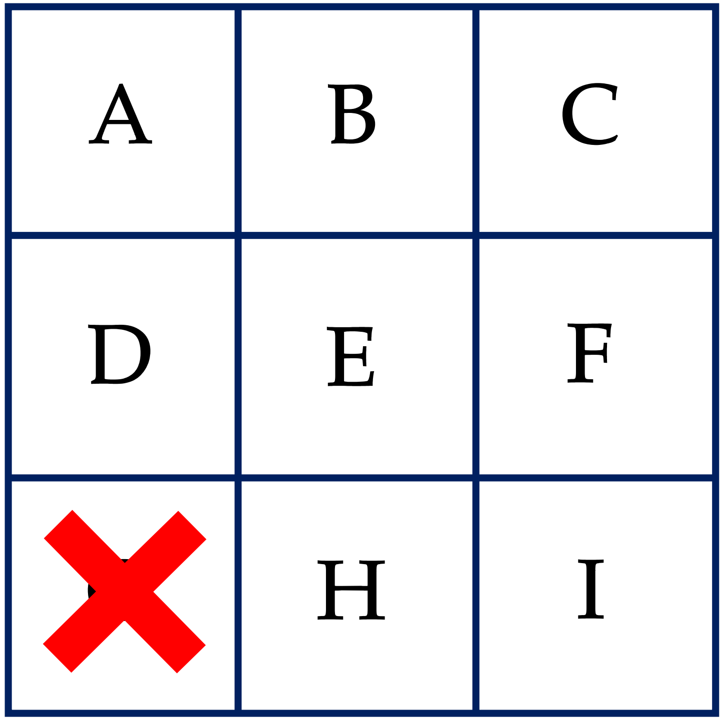
A	B	C
D	E	F
G	H	I

- You start from square A.
- In a move, you are allowed to move up, left, down or right one square given that you stay inside the board.

A	B	C
D	E	F
G	H	I

Let’s start!

MAKE 3 MOVES



I'm sure that you are NOT on square

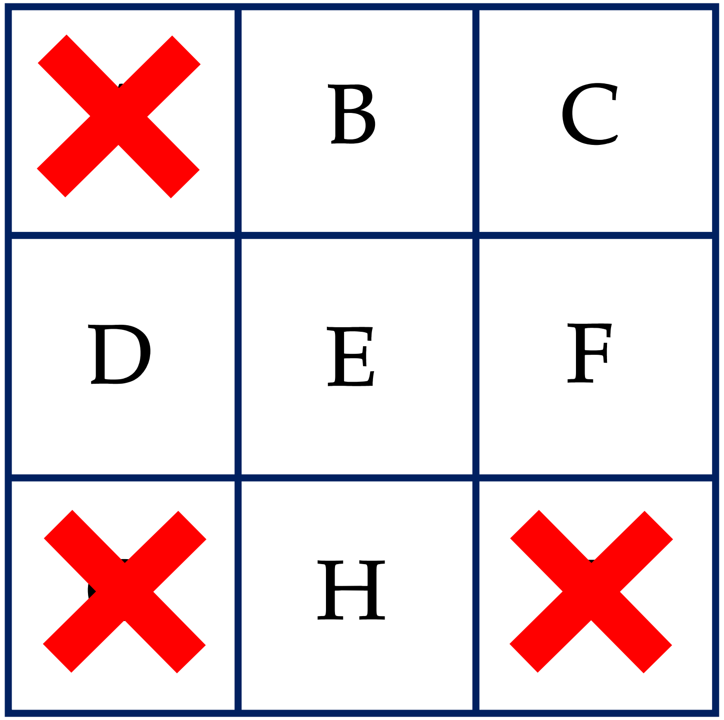
G

Let me destroy it. From now on, you cannot move inside the destroyed squares.

A	B	C
D	E	F
G	H	I

Now,

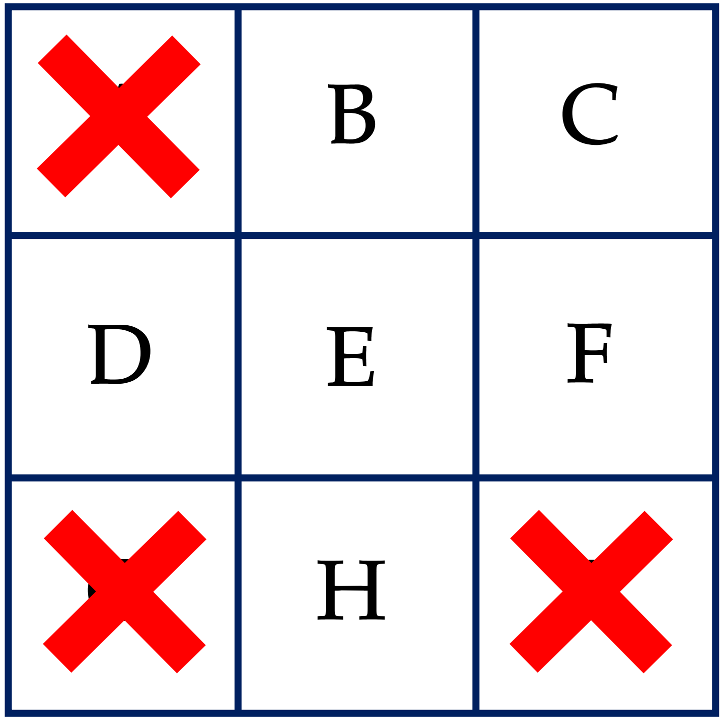
MAKE 2 MOVES



I'm sure that you are NOT on squares

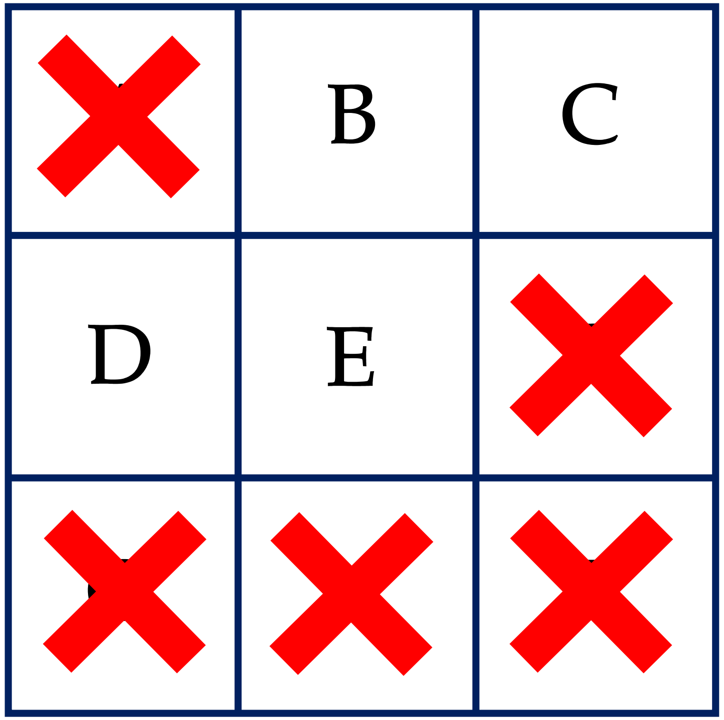
I and A

Let me destroy them.



Now,

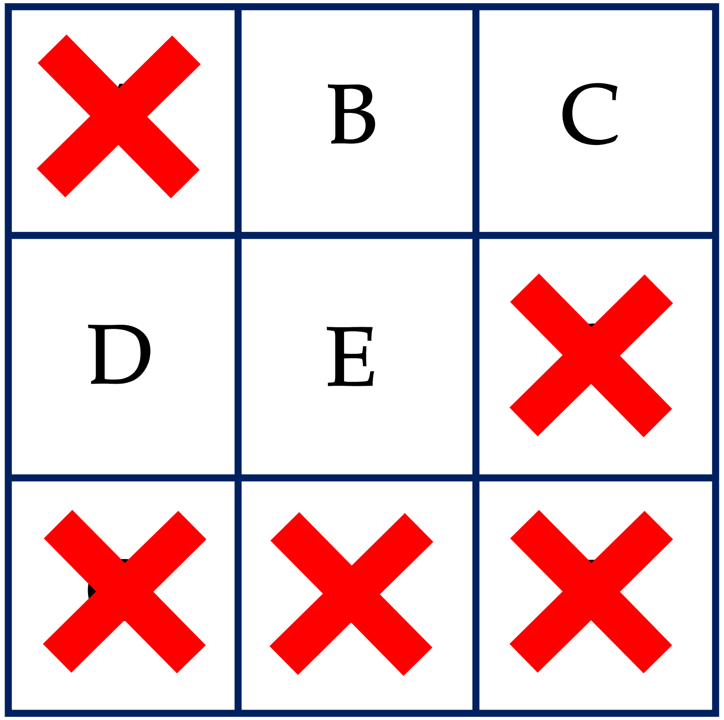
MAKE 1 MOVE



I'm sure that you are NOT on squares

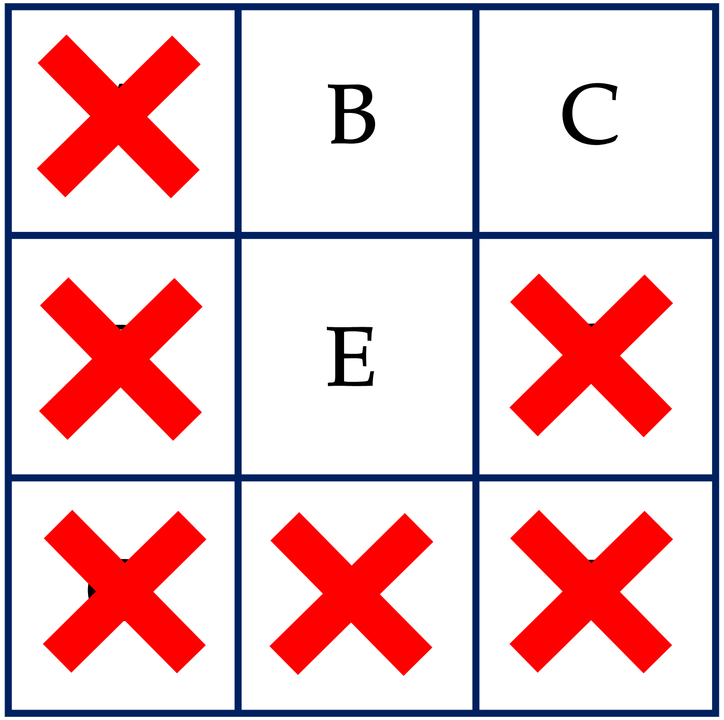
H and F

Let me destroy them.



Now,

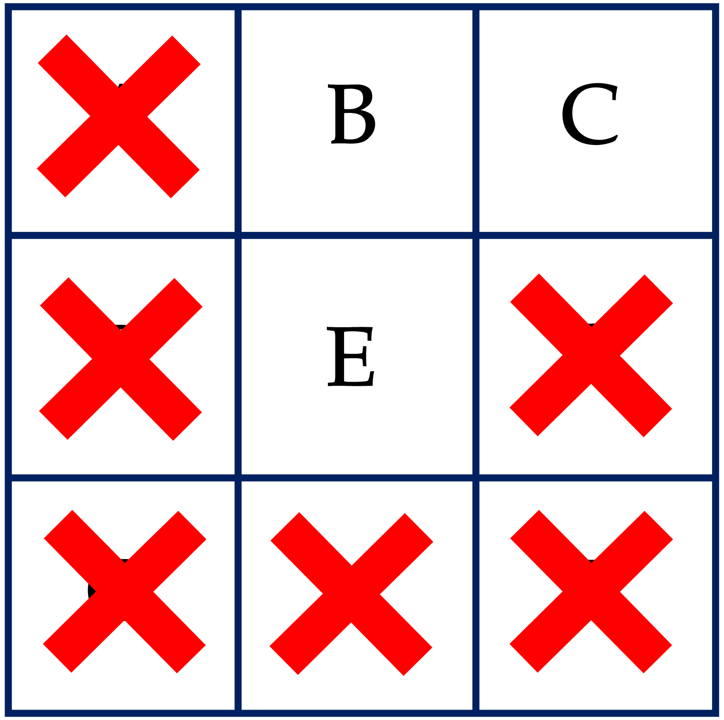
MAKE 2 MOVES



I'm sure that you are NOT on square

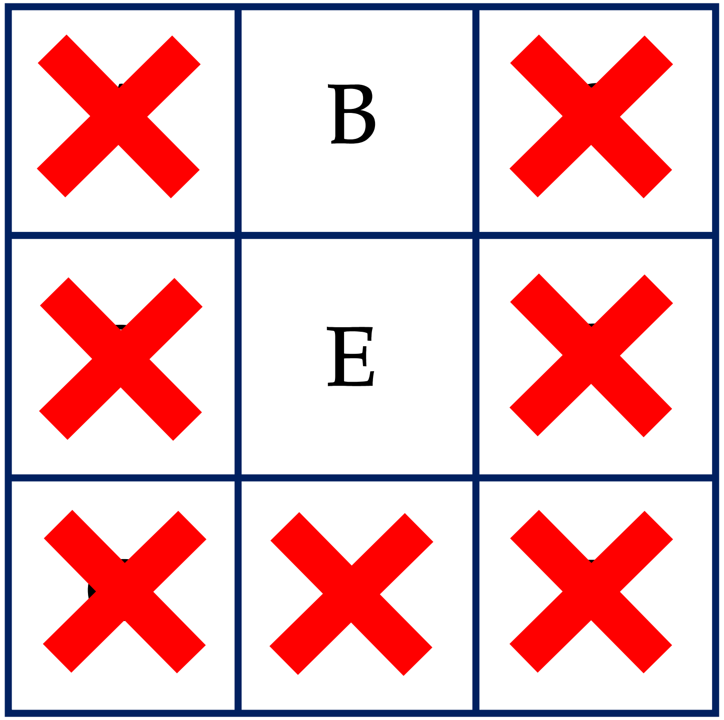
D

Let me destroy it.



Now,

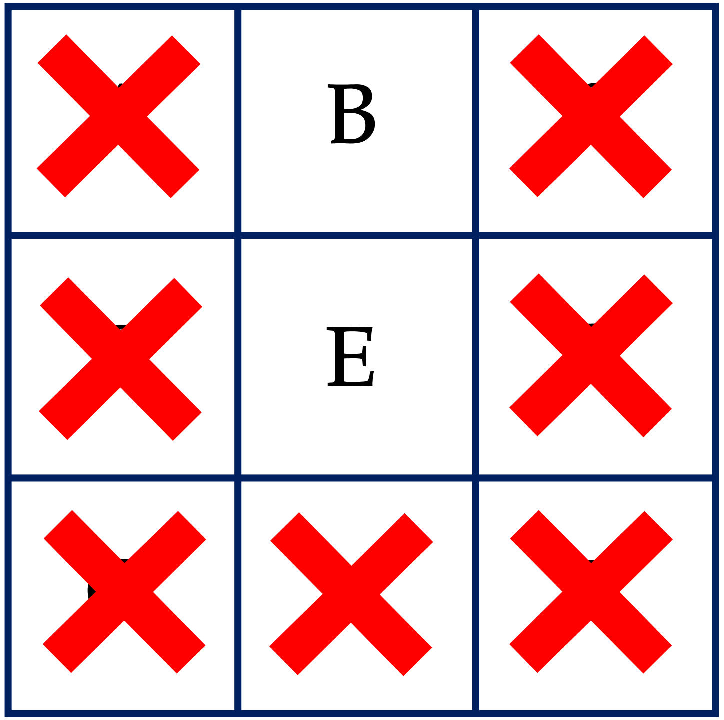
MAKE 5 MOVES



I’m sure that you are NOT on square

C

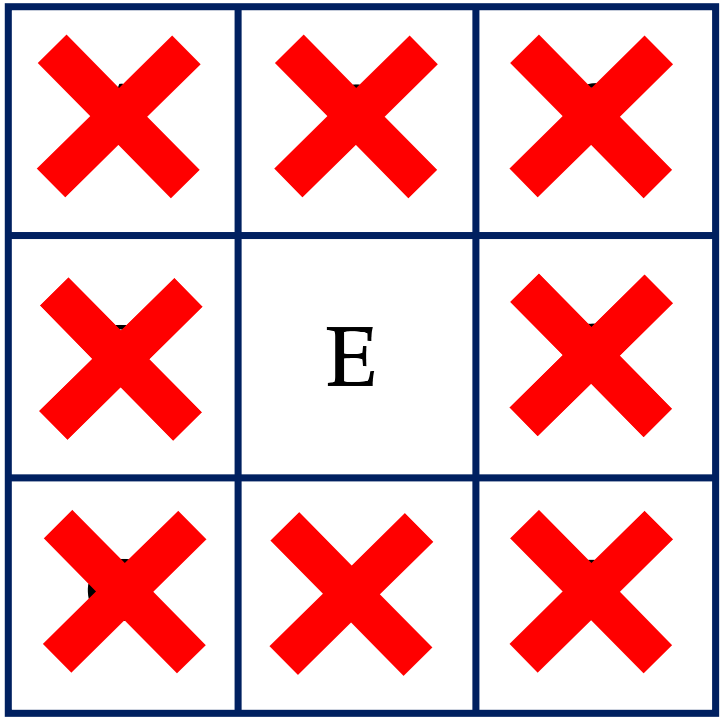
Let me destroy it.



Now,

MAKE 1 MOVE

as your final move.

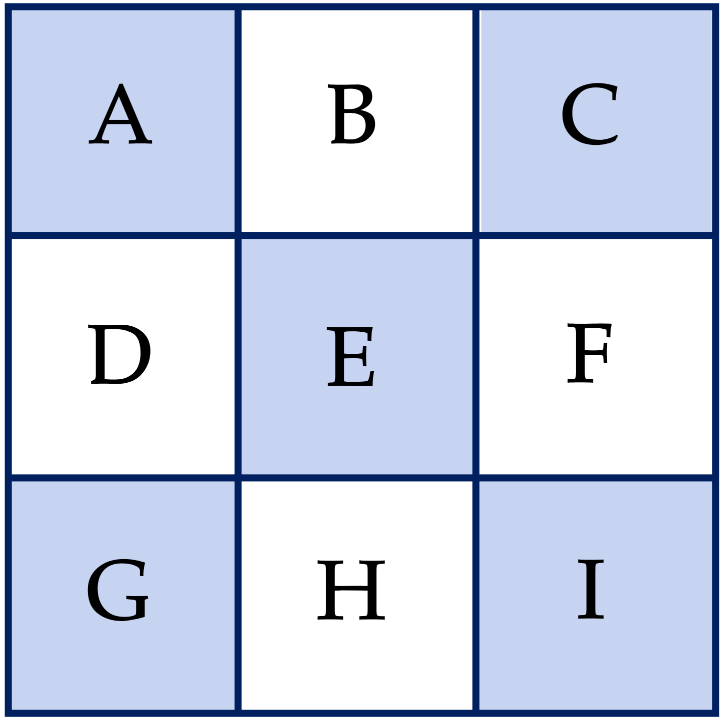


I’m sure that you are NOT on square

B

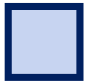
You’re safely standing on square E, aren’t you?

How it is done



First, color the playing board in two colors just like in a chessboard, say using blue and white.

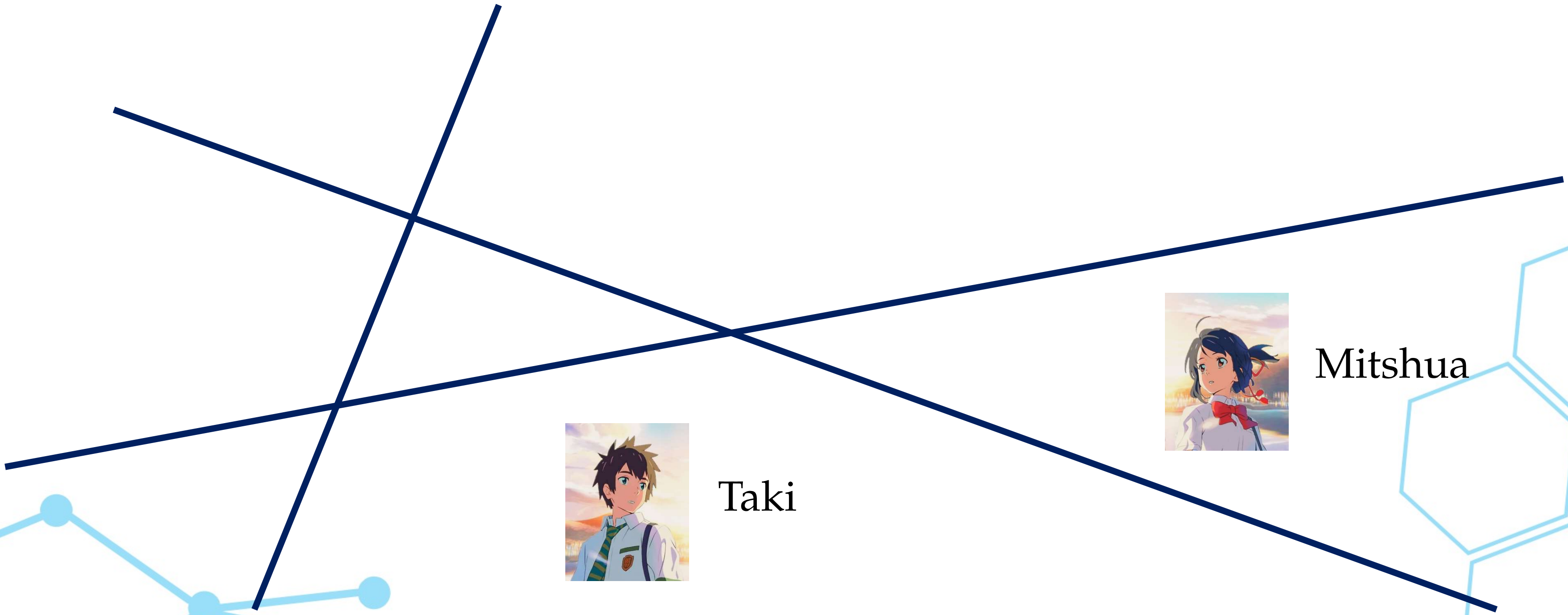
Then,

- Moving from  sends you to  .
- Moving from  sends you to  .

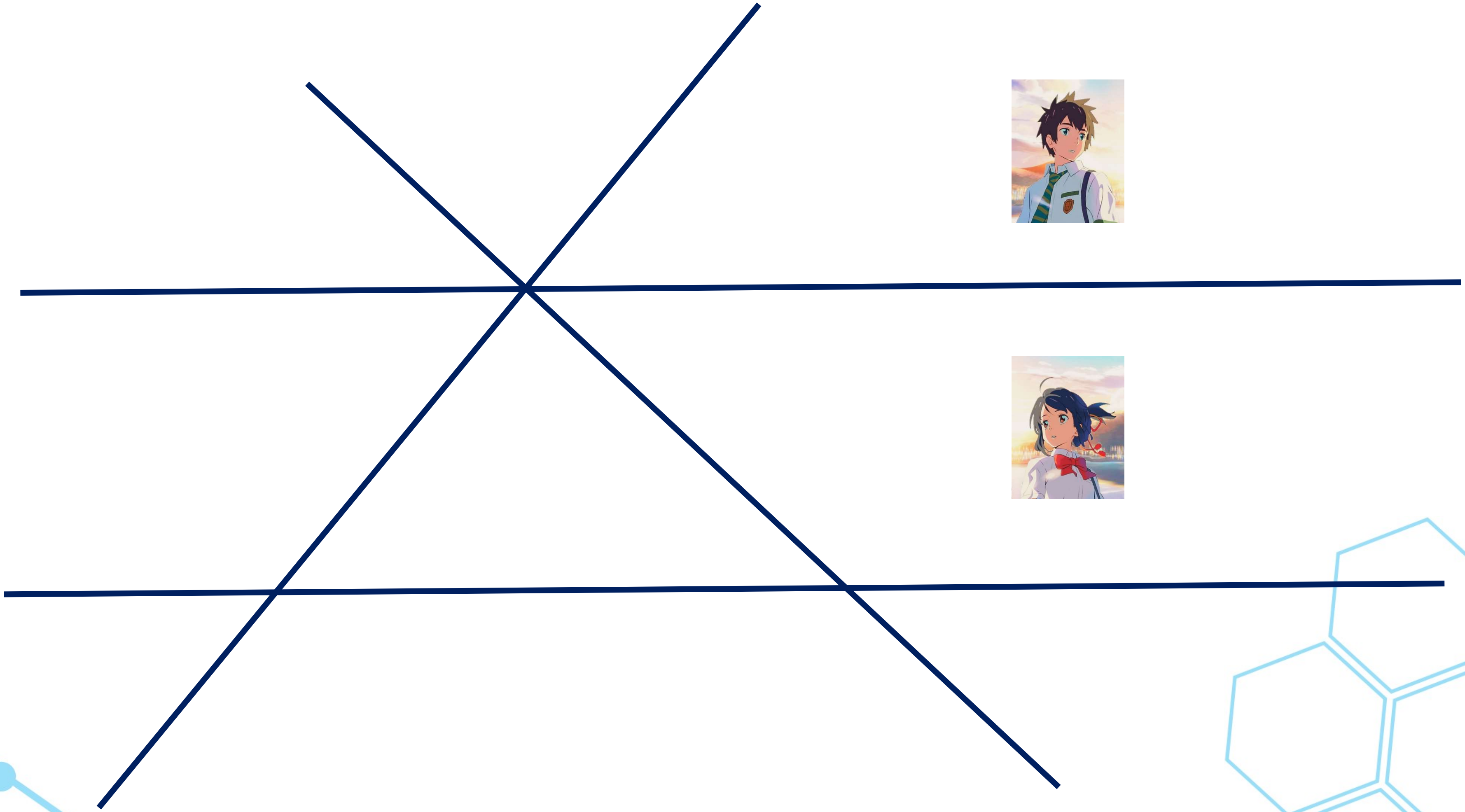
So, I always know the color of your square! I simply destroy the squares of the opposite color.

Mitshua and Taki in a Maze

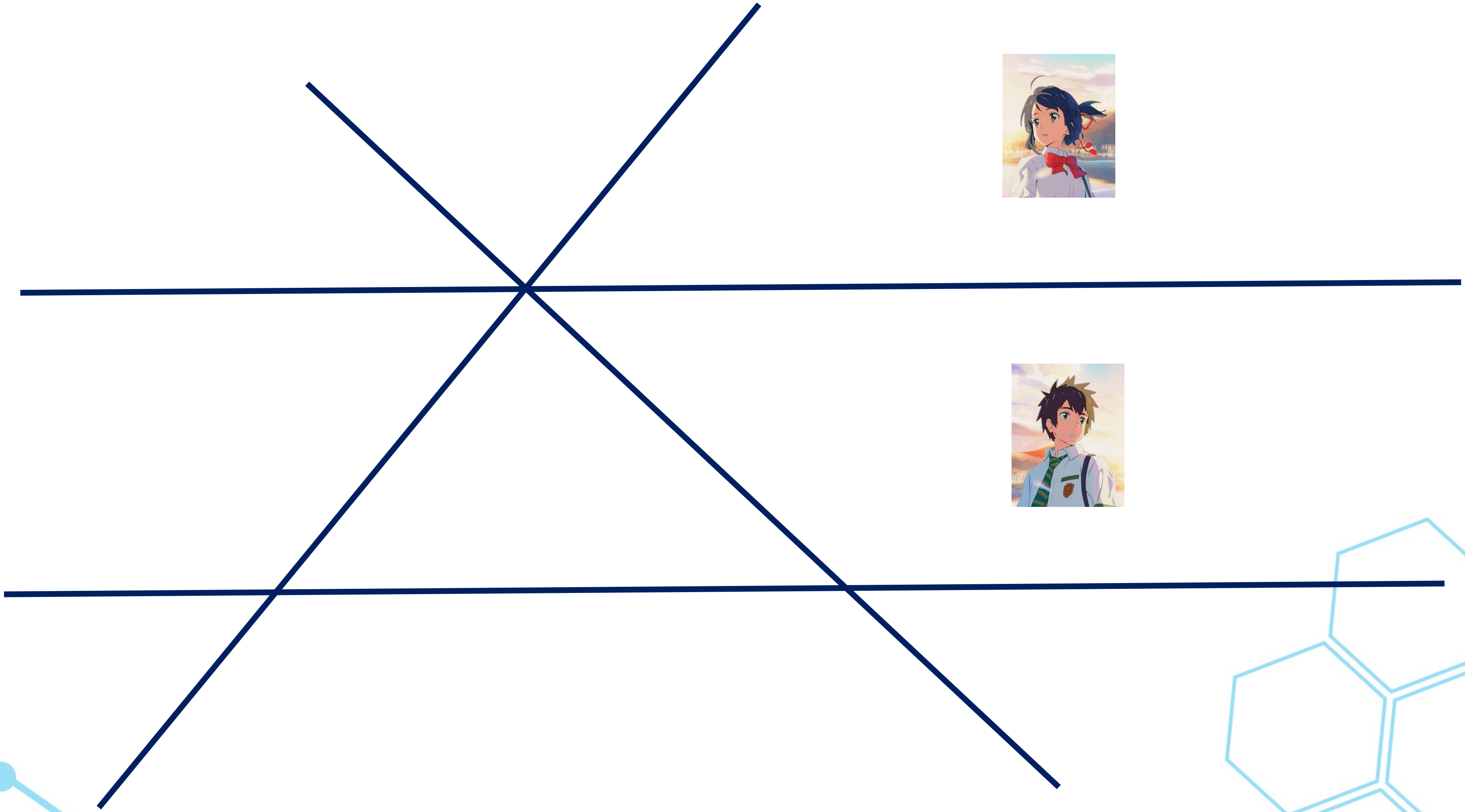
A finite number of straight lines divide the plane into several different connected regions, separated by the line segments. Initially, Mitshua and Taki are standing in the regions that share an edge. They are allowed to walk into different regions sharing an edge, but when they do this, they both have to do it simultaneously. Is it possible that Mitshua and Taki meet in the same region?



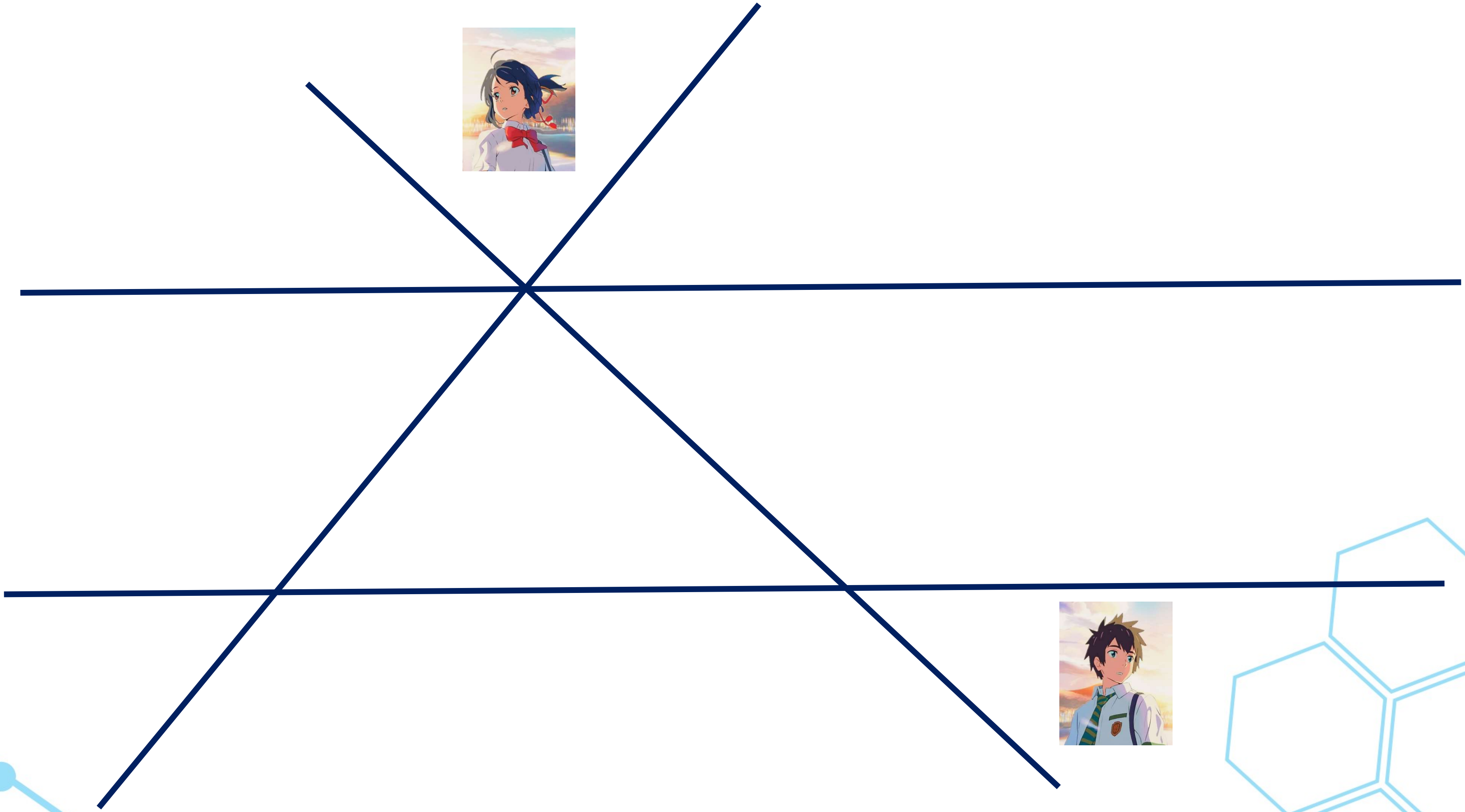
Let's try an example…



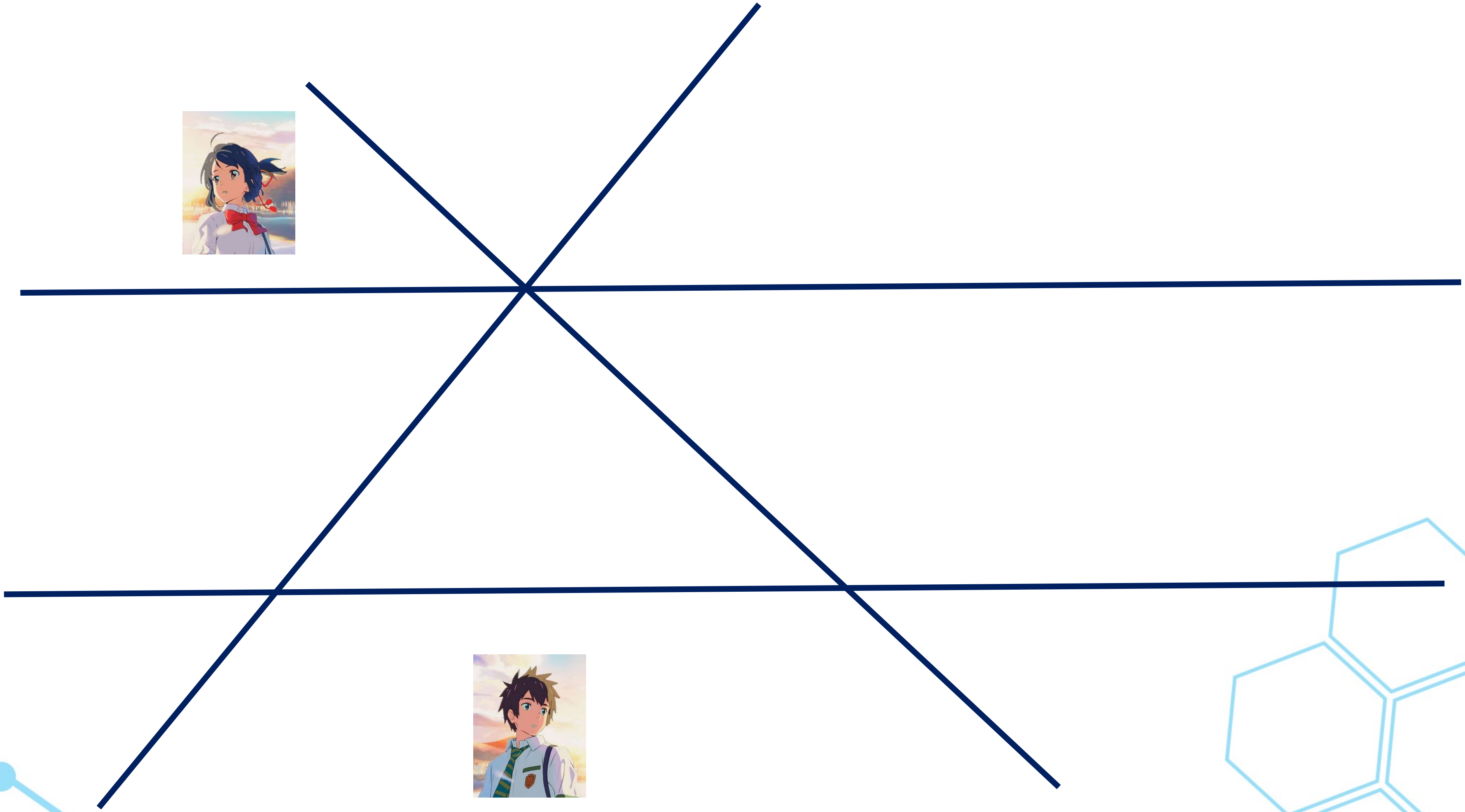
Let’s try an example…



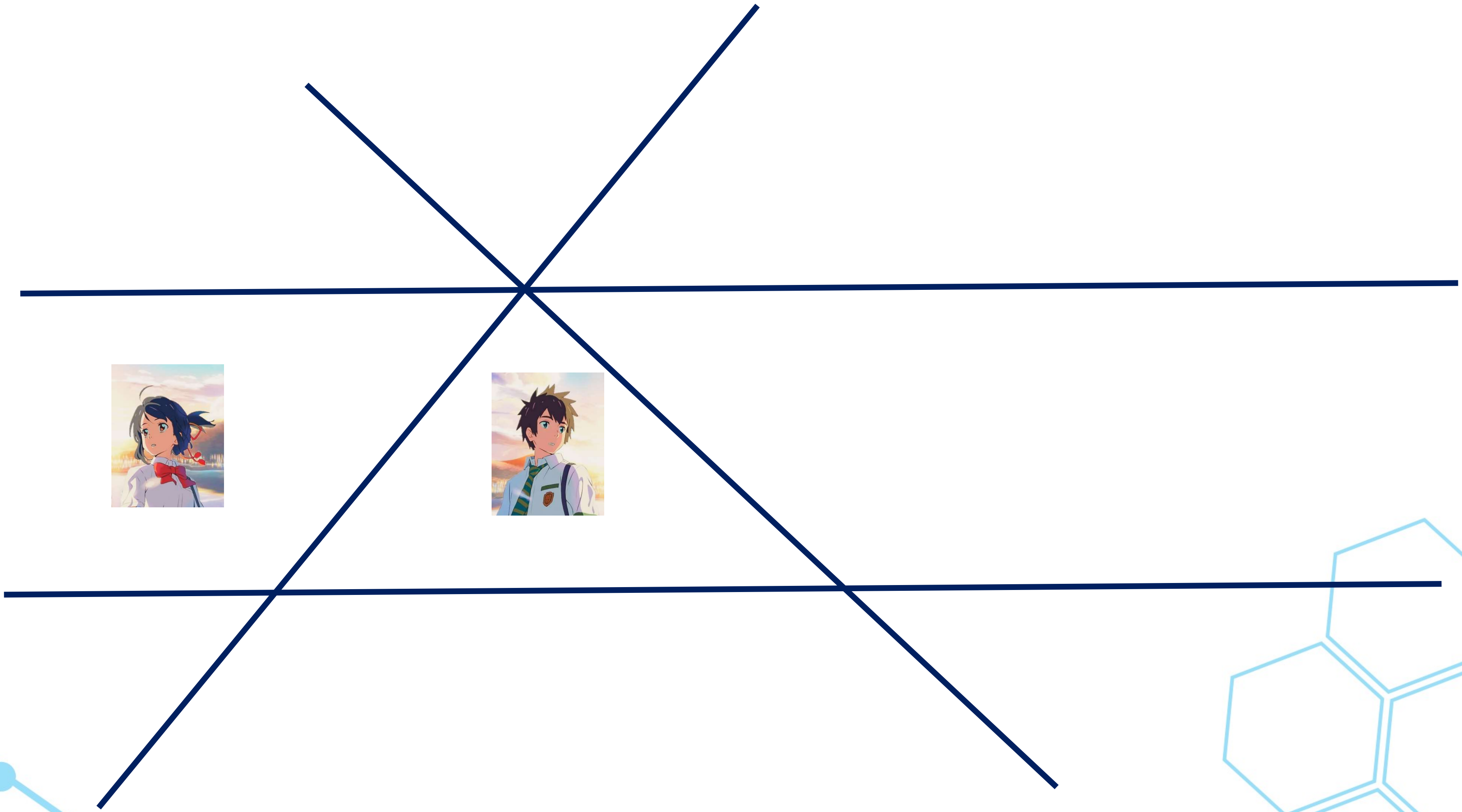
Let’s try an example…



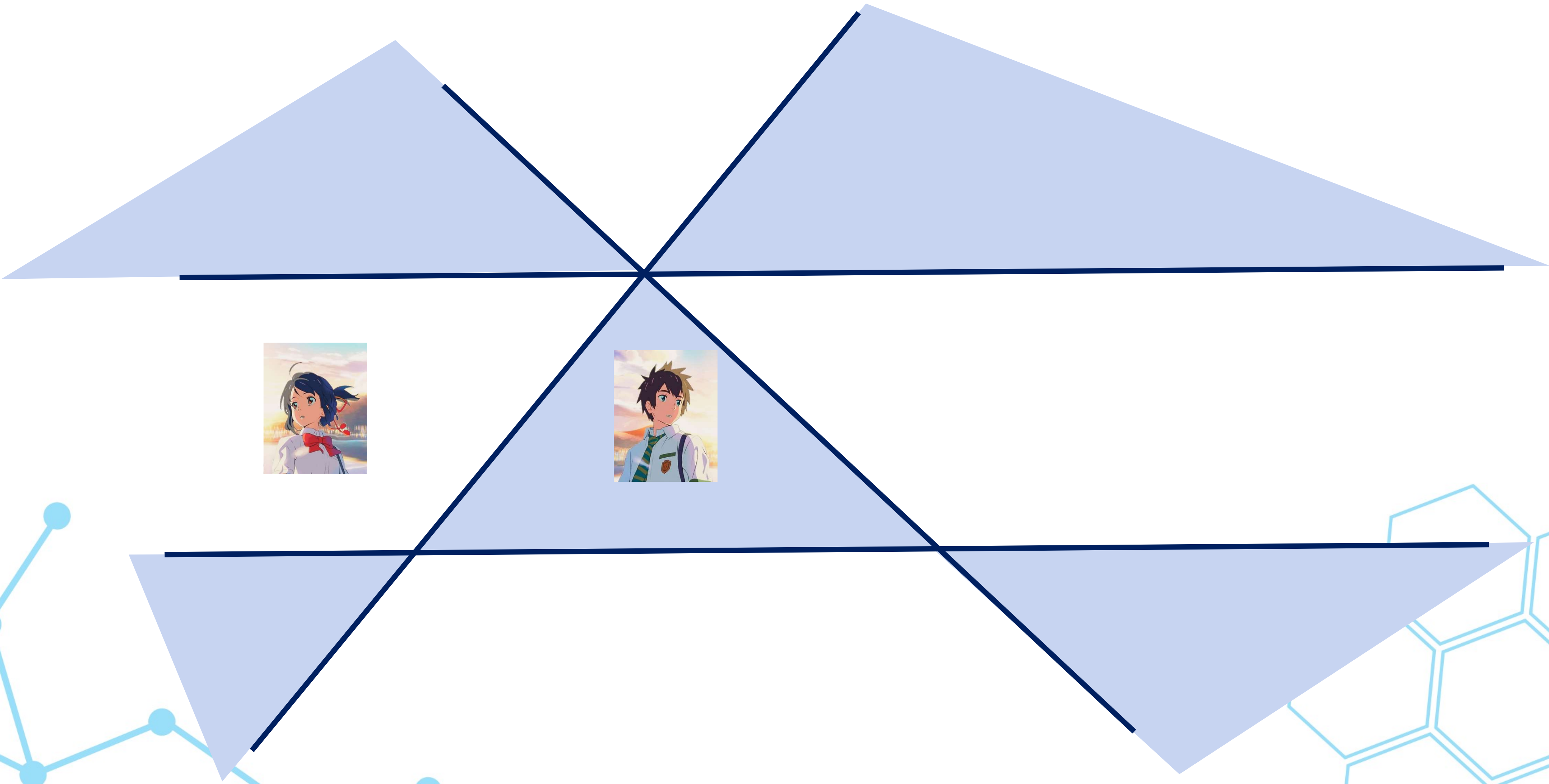
Let’s try an example…



Let's try an example…



Just look at this picture…

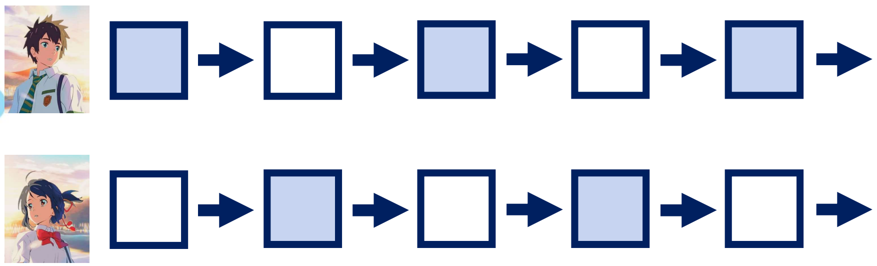


Half a solution

- Color the regions in blue and white so that regions sharing an edge receive different colors.

Is this possible in general?

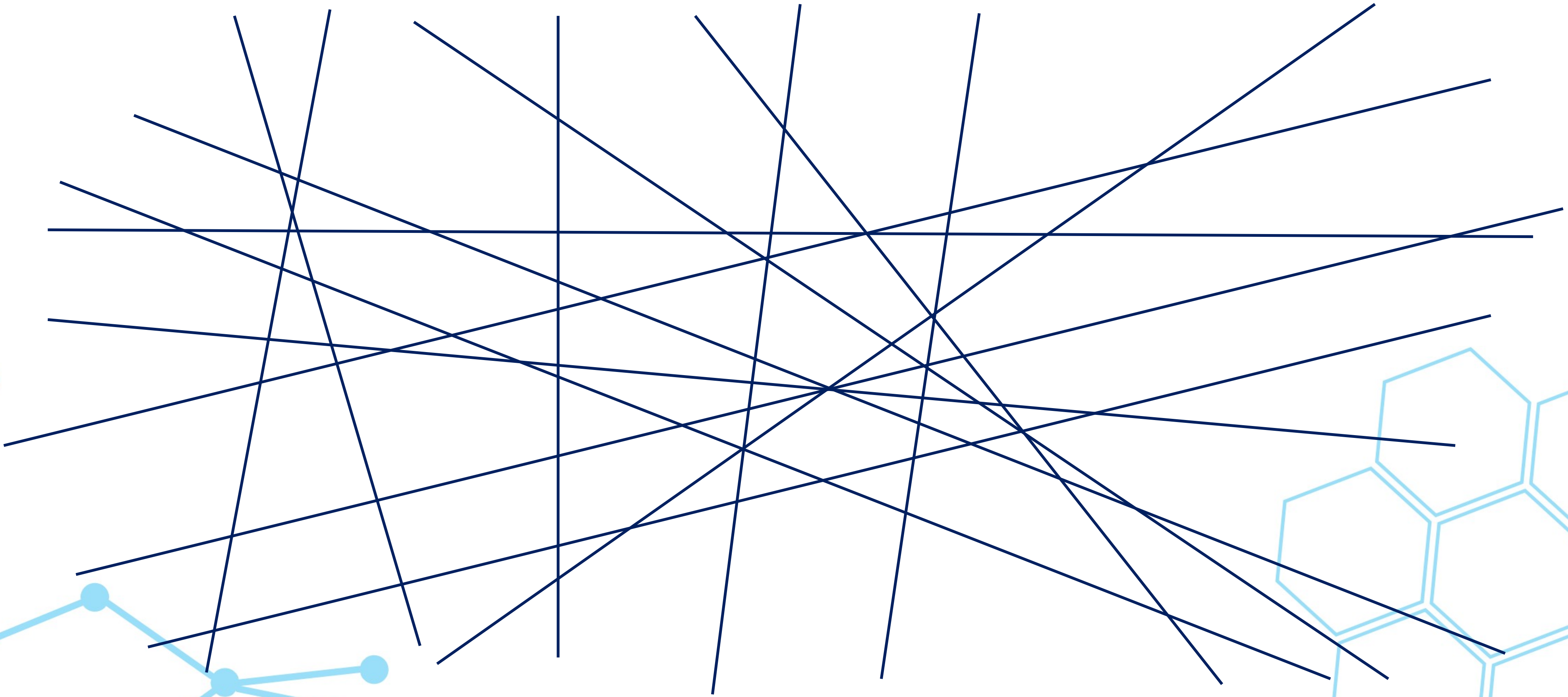
- Then, Mitshua and Taki's starting colors are different.
- Each of them alternates between blue and white regions and thus their colors will always be different.



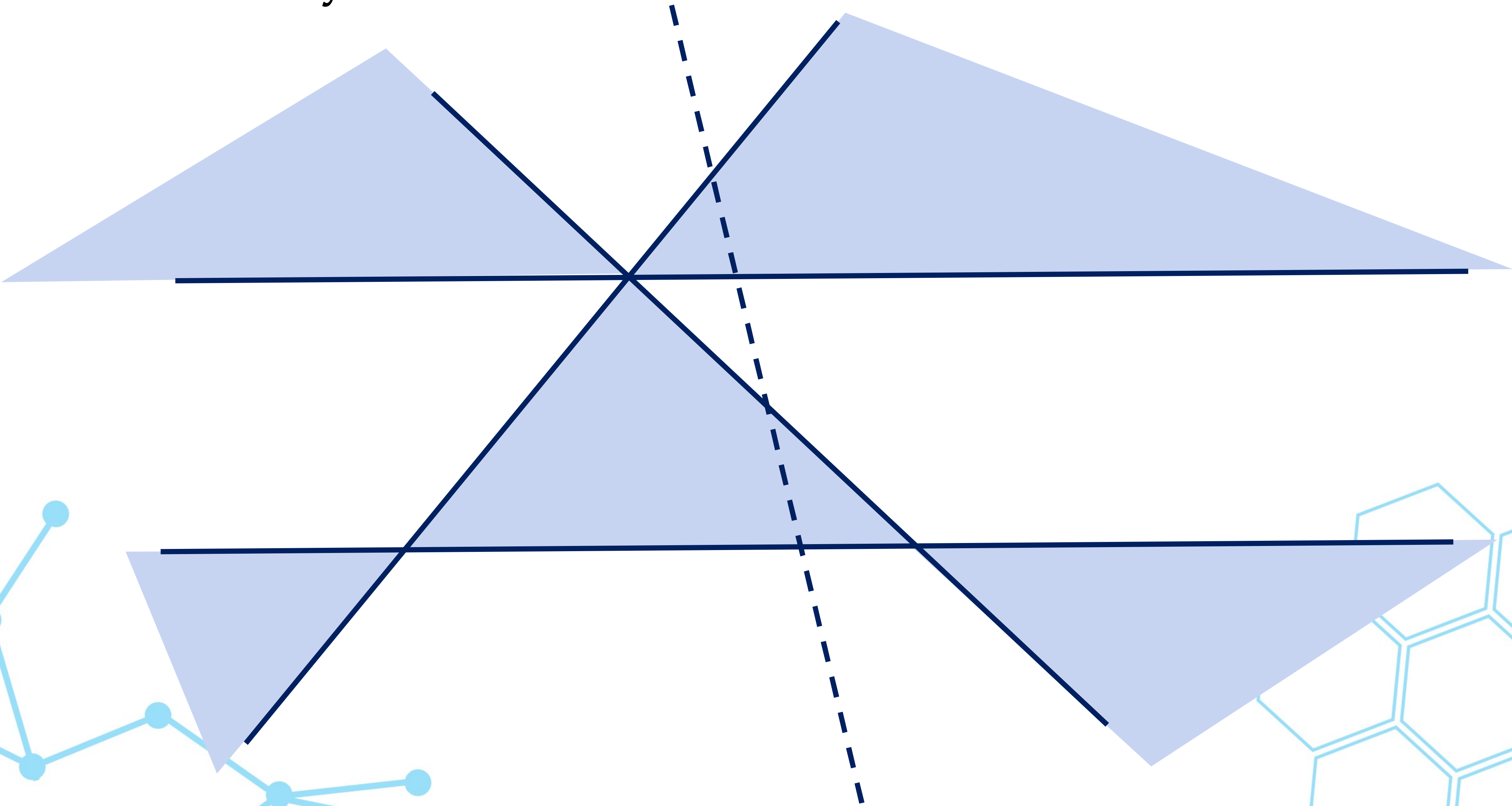
- So, they cannot meet each other, THE END.

Another half is not trivial

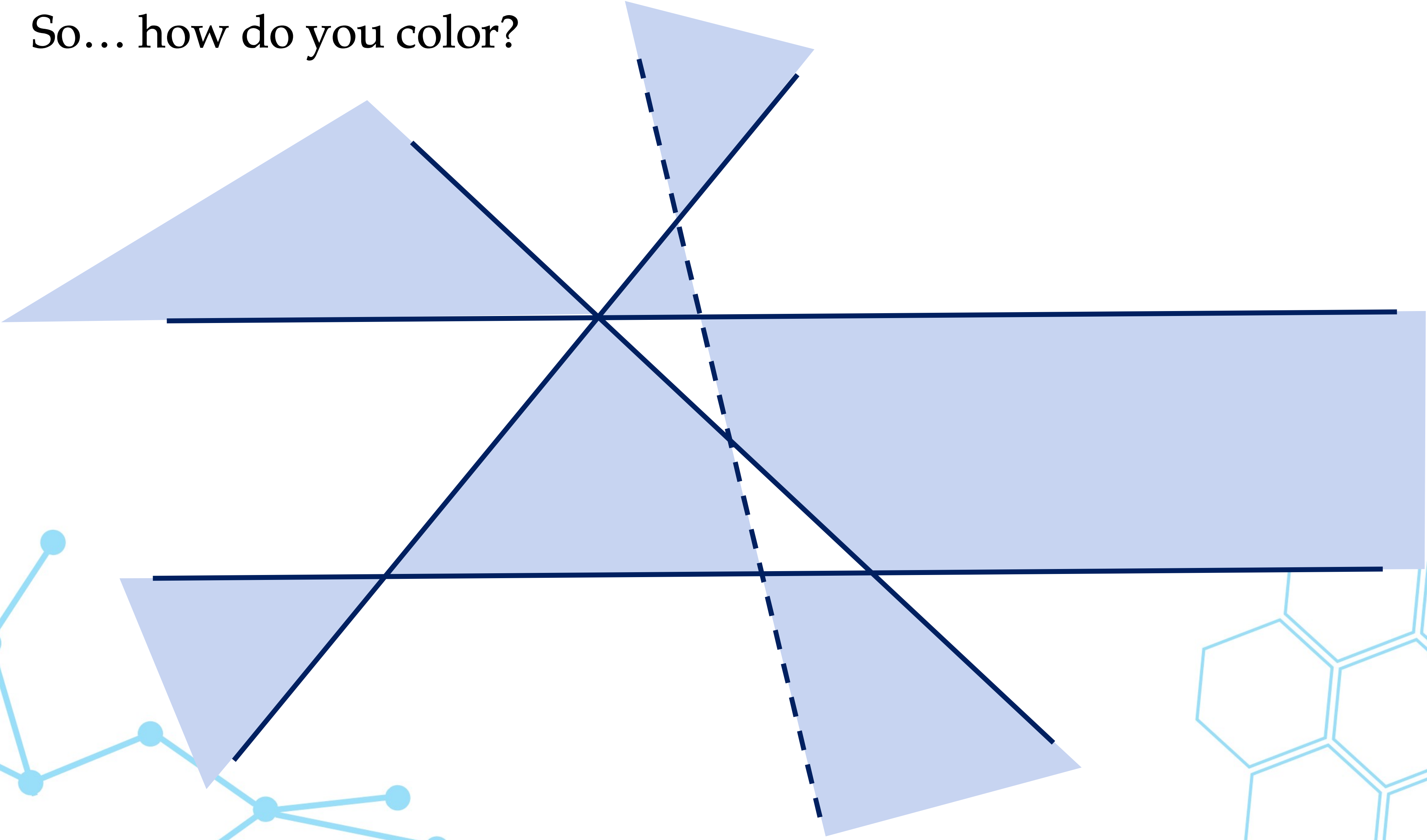
How would you color the following configuration? What about a configuration with 10^{100} lines?



So… how do you color?

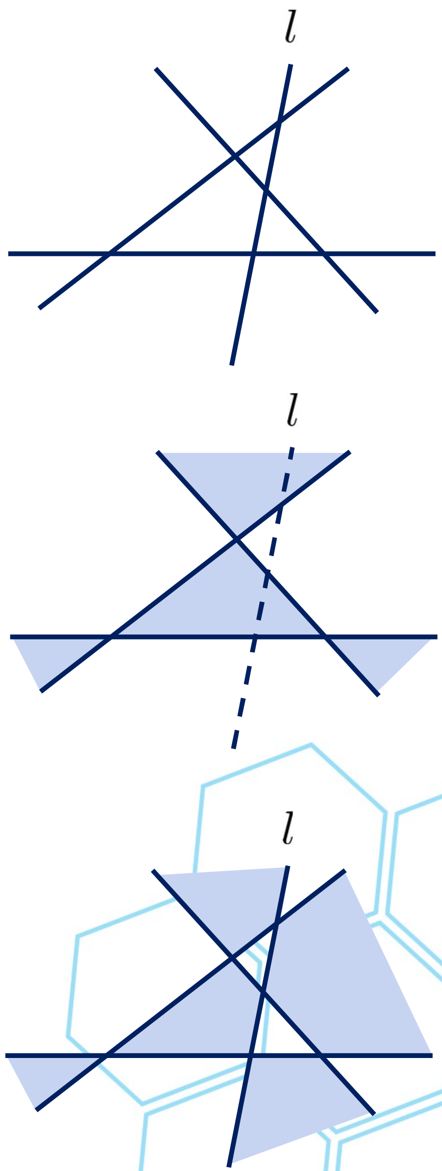


So… how do you color?



Induction…

- If there is only one line, coloring is easy!
- Say we have proved that a coloring exists for $1, 2, \ldots, n-1$ lines.
- Consider an arbitrary configuration with n lines.
- Pick one of the lines, say l and remove it. Then, resultant picture can be 2-colored.
- Now, put l back into the picture and switch the color of every region to the "right" of l.
- Then, if two regions are adjacent via l, they have different colors. If they are adjacent via something other than l, then they also have different colors by inductive coloring.

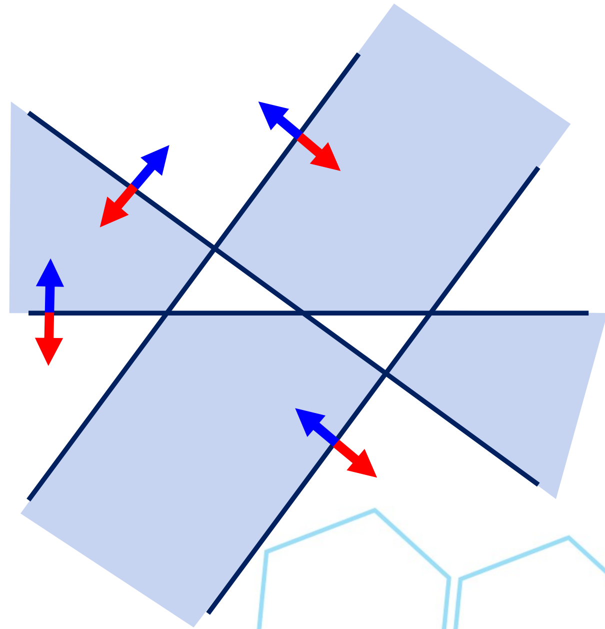


Without Induction

- For each line l, define one of the half-planes associated with l to be good and the other bad.

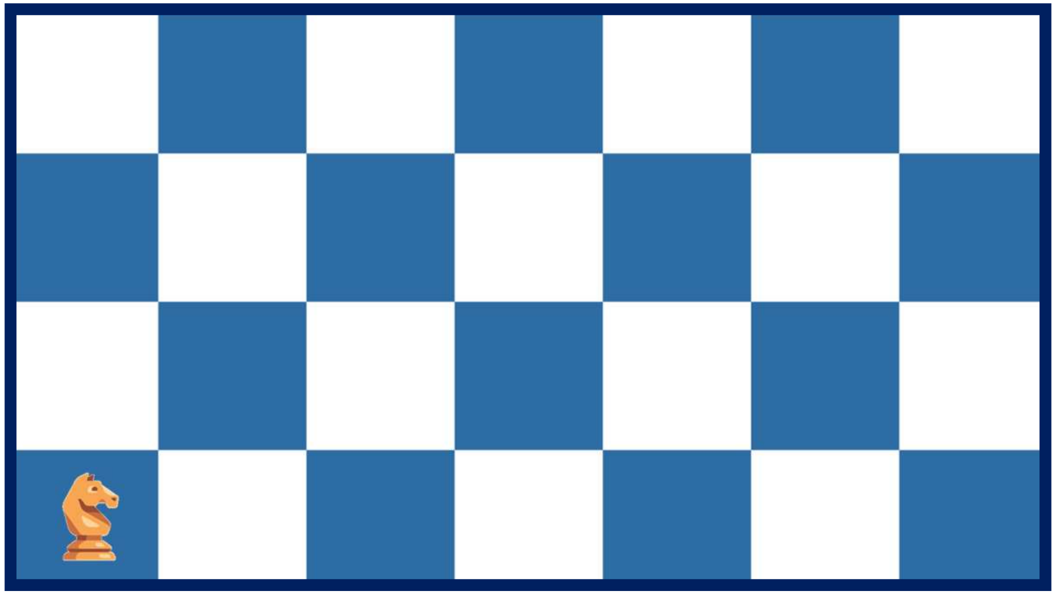
- Color each region blue if it lies in odd number of good half-planes and blue otherwise.

- Then, whenever there are two adjacent regions separated by line l, one region will be in good half-plane of l and the other will be in bad half-plane. "Goodness" of other half-planes remain the same. Hence, parity of number of good half-planes changes.

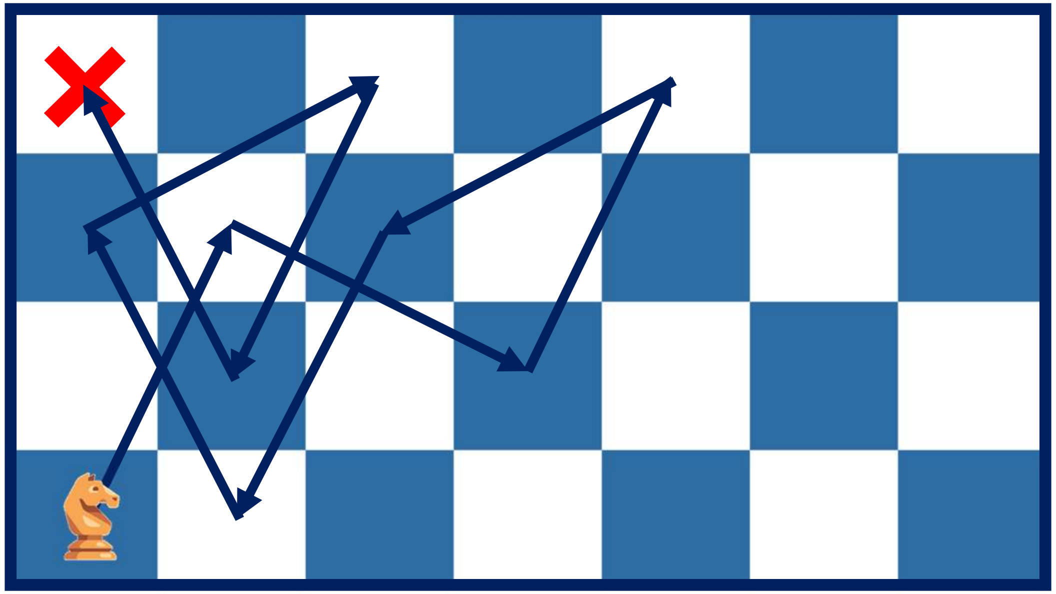


Knight's Tour on a Strip

Let n be a positive integer, a chess knight is placed at the lower left corner of a $4 \times n$ board. For which n will it be possible to move the knight so that it visits all the squares exactly once and can return to the lower left corner at the $4n$-th move?

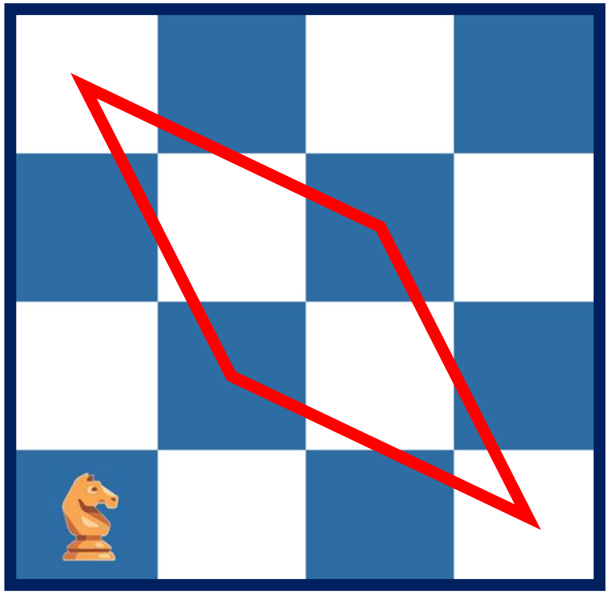


Example



Quick Exercise

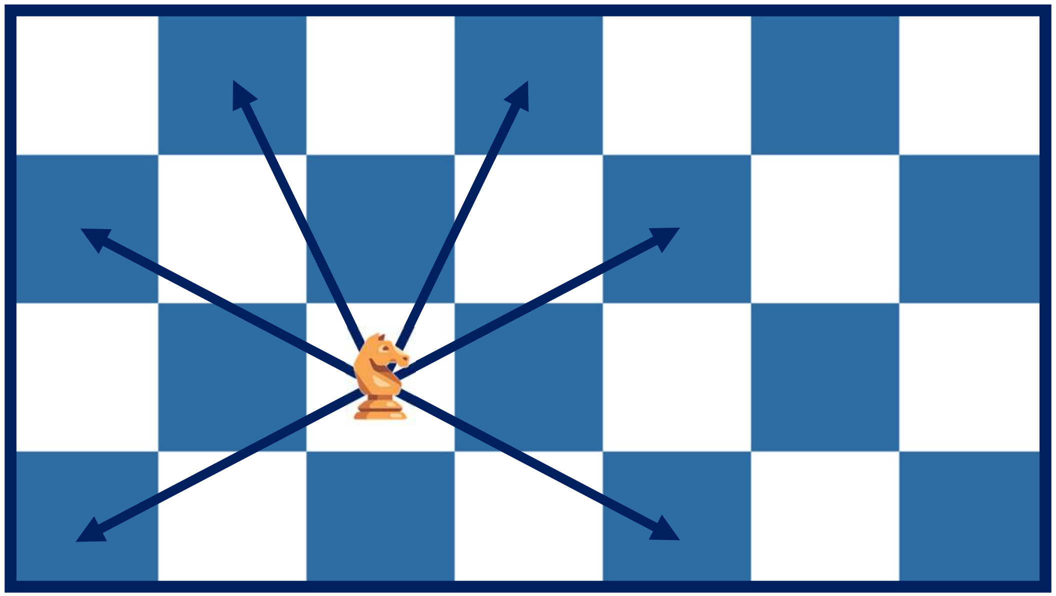
Try to solve the problem for $n = 4$. There is an easy solution.



Immediate observations

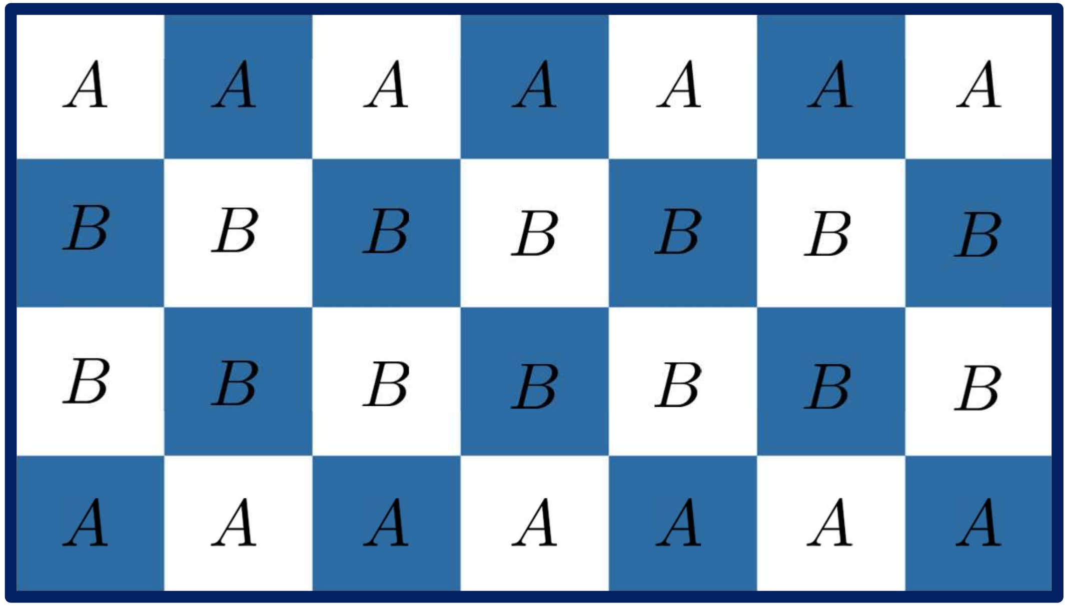
“Observation”: Journey of length $4n$, without repeating the squares basically means visit every square exactly once.

Observation 1: If we colour the squares white and blue, then the knight alternates colours.



A Subtle Observation

Observation 2: Use another 2 colours A and B, then colour the squares like this:



Then, colour A will always be followed by colour B.

Solution

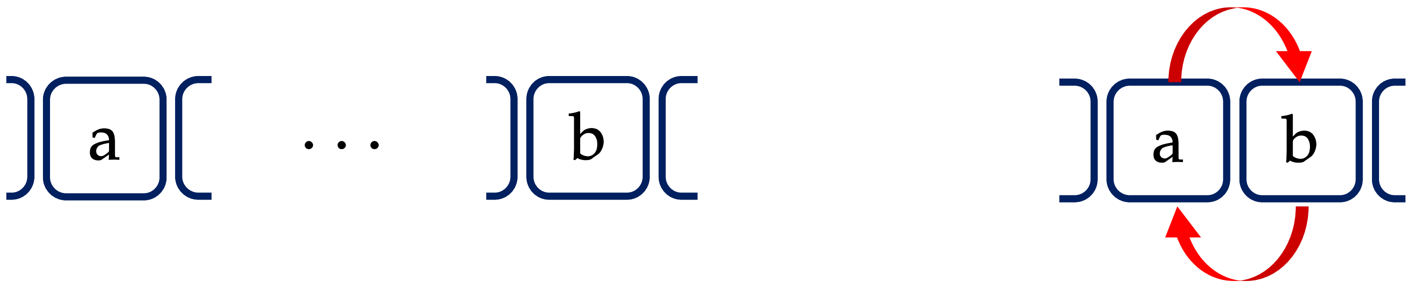
- Suppose to the contrary that the knight can tour around the board.
- Then, $A \to B$ implies that A and B are alternating (why).
- On the other hand, the knight alternates between white and blue.
- Thus, the knight can only visit at exactly two of the following types of squares.

Parity of Permutation

Suppose we have n cards numbered $1, 2, \ldots, n$ arranged in a row in some order.

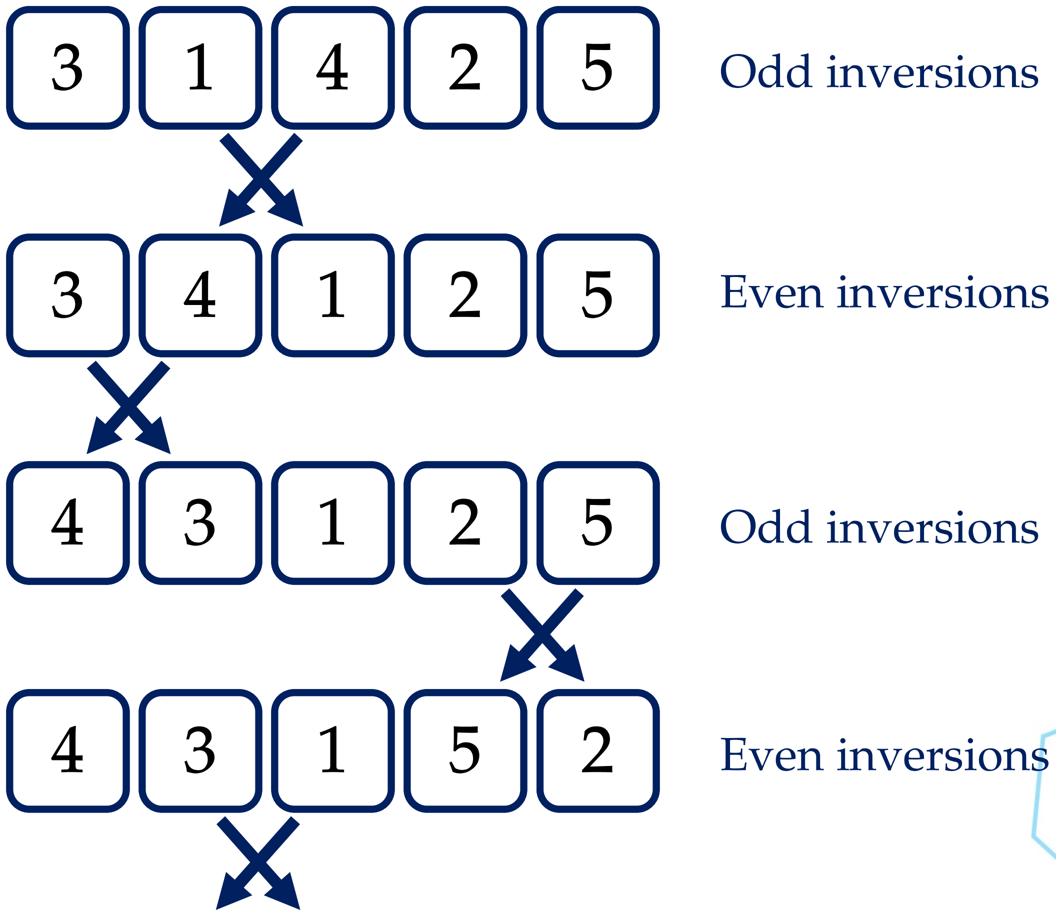
Recall: A pair of cards is called inverted if the larger card is on the left.

Recall: Number of inversions decrease by 1 when you swap 2 consecutive inverted pair of cards.

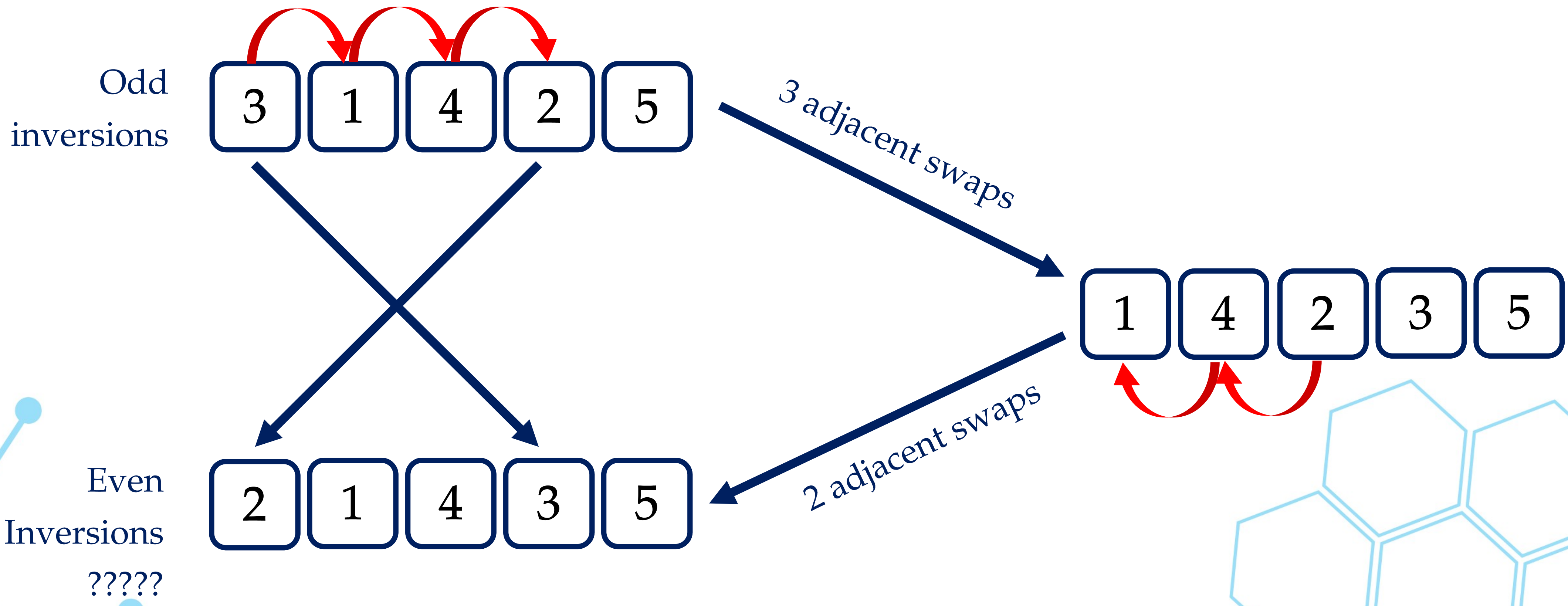


By symmetry, number of inversions increase by 1 when you swap 2 consecutive non-inverted pair of cards. Thus, parity of the number of inversions ALWAYS change!

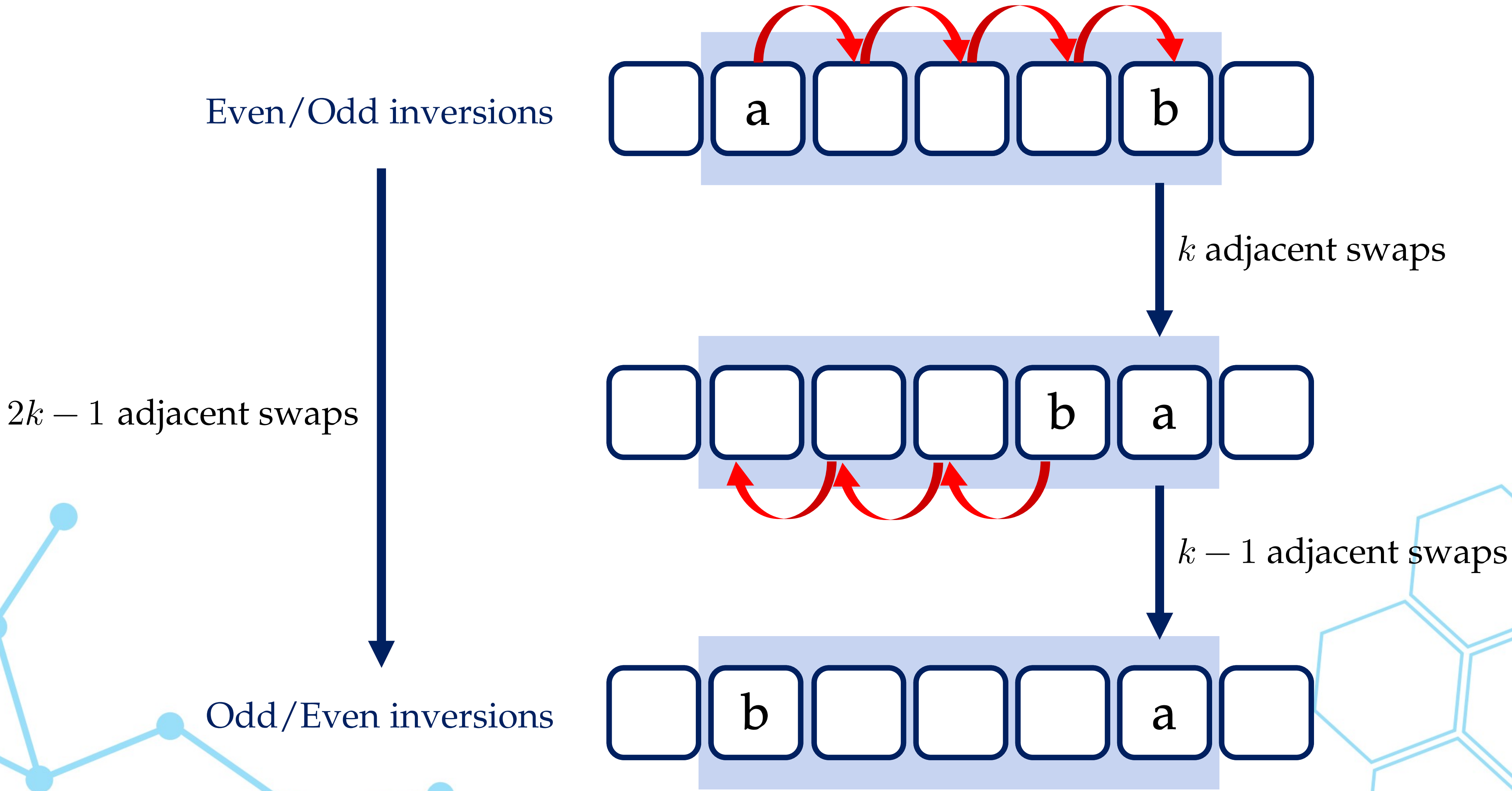
Example


3 1 4 2 5
Odd inversions
3 4 1 2 5
Even inversions
4 3 1 2 5
Odd inversions
4 3 1 5 2
Even inversions

What about arbitrary swaps?



General Case



Permutations and Swaps

Call a permutation even if it has even number of inversions. Call it odd otherwise.

Then, the permutations fall into two disjoint groups and we change groups by swapping any two entries.