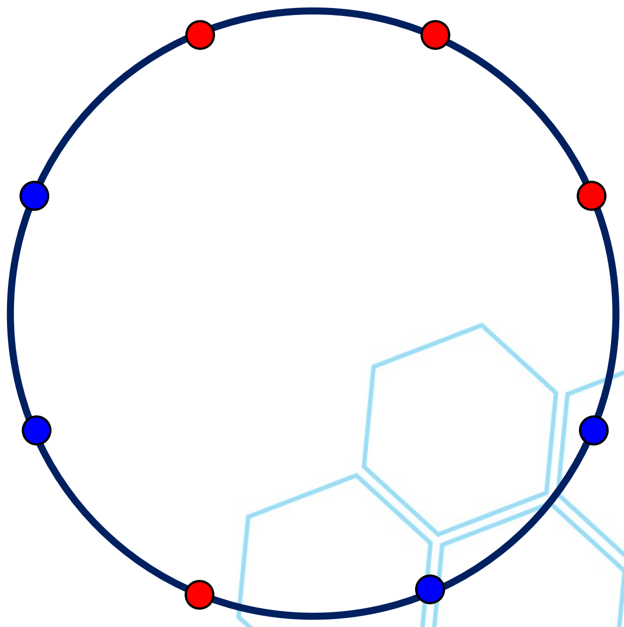
Let’s start at 1:05

Read this problem while we wait…

Consider $2n$ points equally spaced around a circle. Suppose that n of the points are coloured blue and the remaining n points red. We write down the distance between each pair of blue points in a list, from shortest to longest. We similarly write down the distance between each pair of red points in another list, from shortest to longest. Prove that the two lists of distances are identical (note that the same distance may occur more than once in a list).



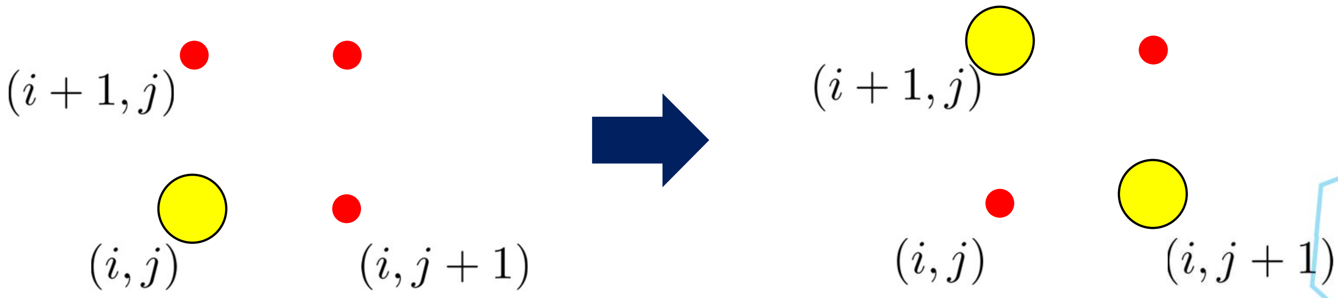
Lecture – 2

Invariants

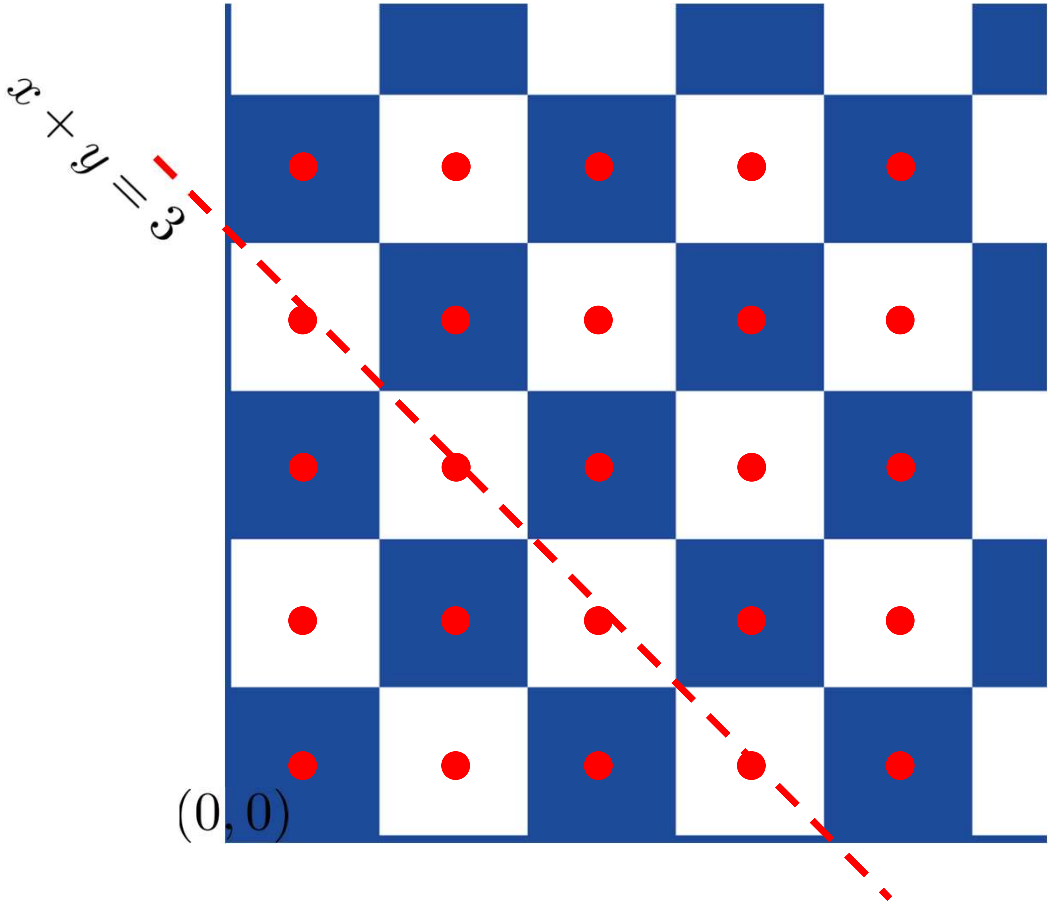
Pebbling Game

A game of pebbles is going to be played in the first quadrant of the coordinate plane. Initially, there is one pebble at the origin. In a move, Botez can remove a pebble from (i, j) and place one pebble each on $(i+1, j)$ and $(i, j+1)$, provided that (i, j) had a pebble to begin with and that $(i+1, j)$ and $(i, j+1)$ did not have pebbles.

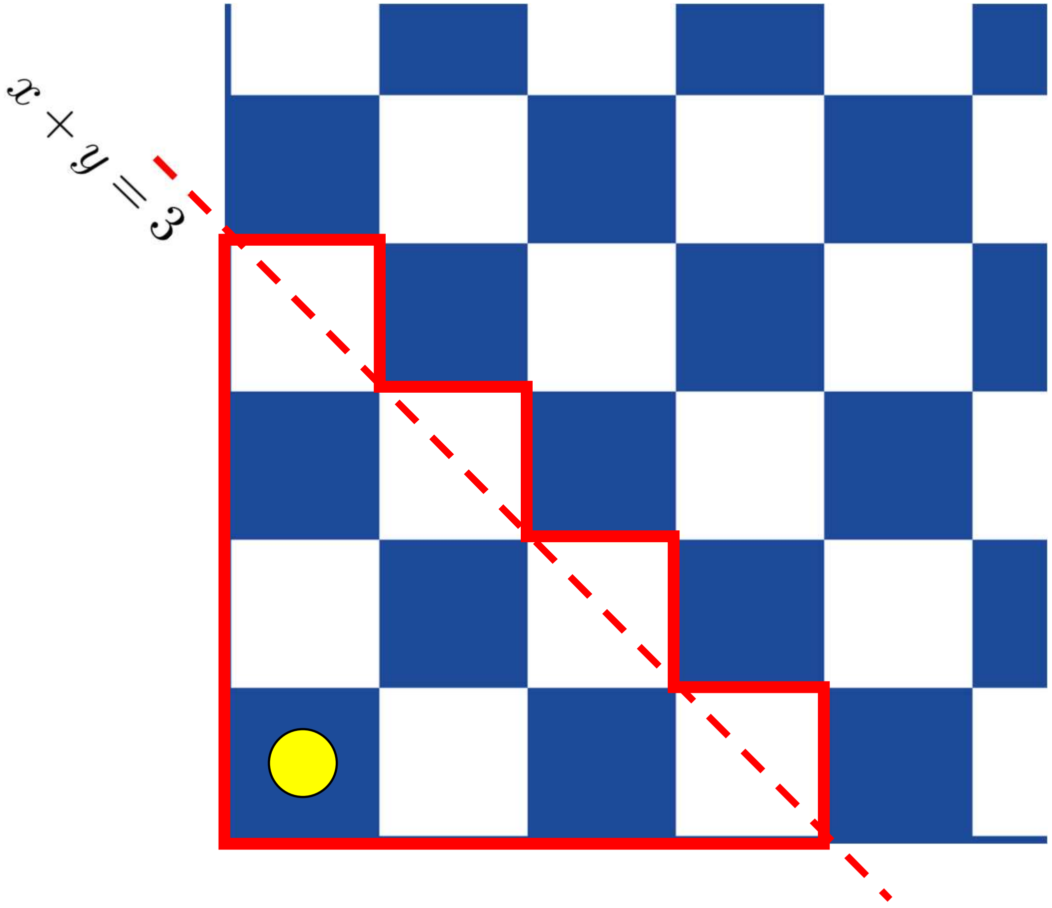
Prove that at any point in the game, there will be a pebble below or on the line $x + y = 3$.



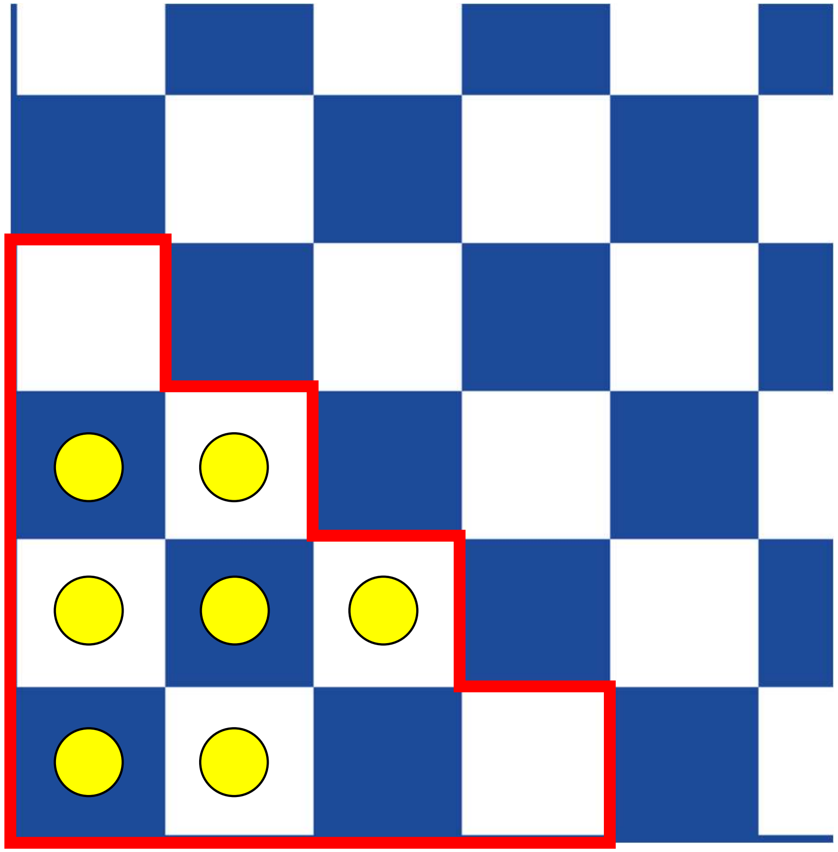
Example



Example

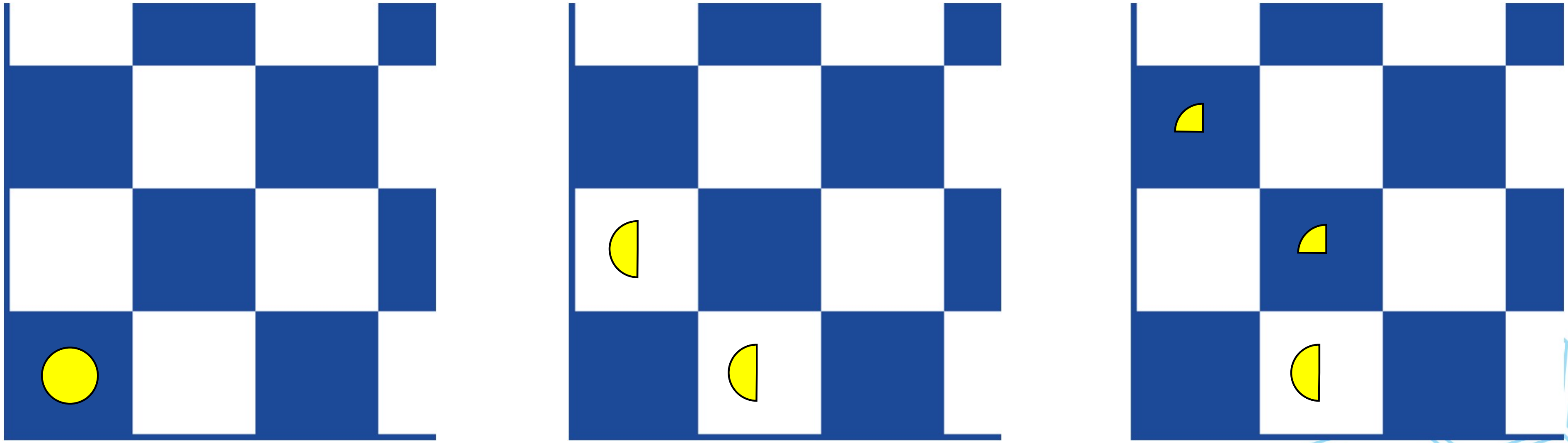


Example



Main Idea

Instead of removing one pebble P and place two pebbles, think of breaking P in half and put the resulting pieces in place of the two pebbles.



Then, total "weight" of the pebbles on the board will NOT change!!

Leveraging the Main Idea

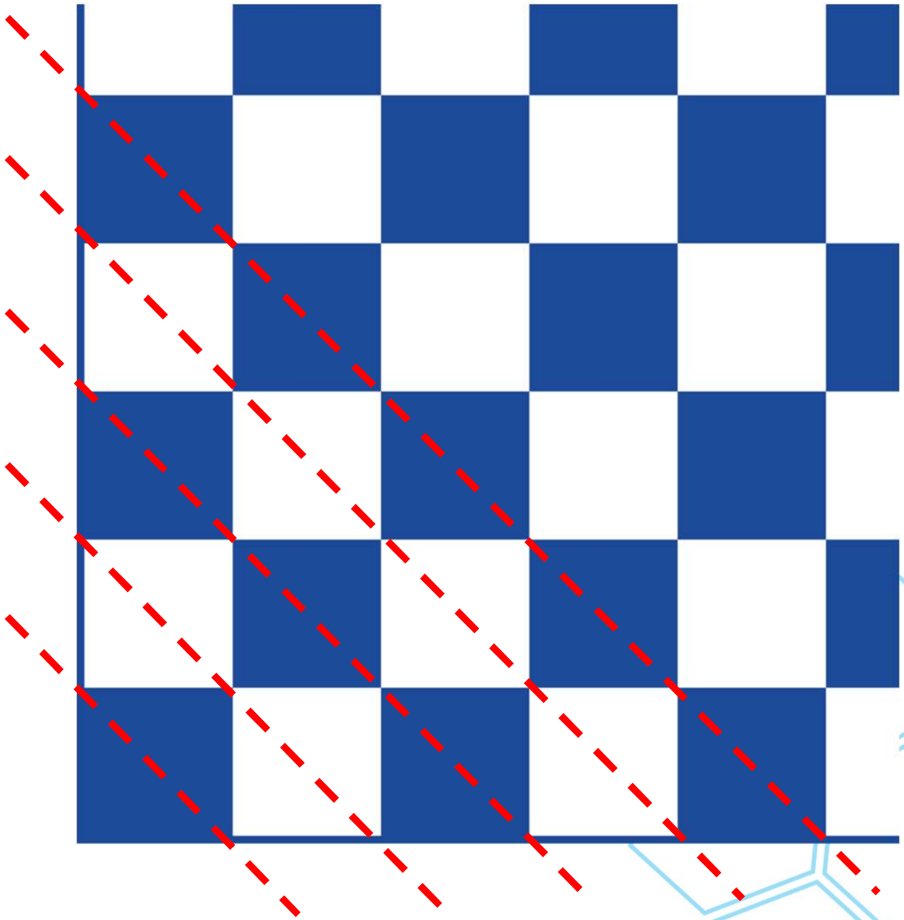
Pebbles on this line have weight 1/16

Pebbles on this line have weight 1/8

Pebbles on this line have weight 1/4

Pebbles on this line have weight 1/2

Suppose initial pebble has weight 1



Pebbles on line $x + y = N$ have weight $1/2^N$.

Solution

We started with weight = 1.

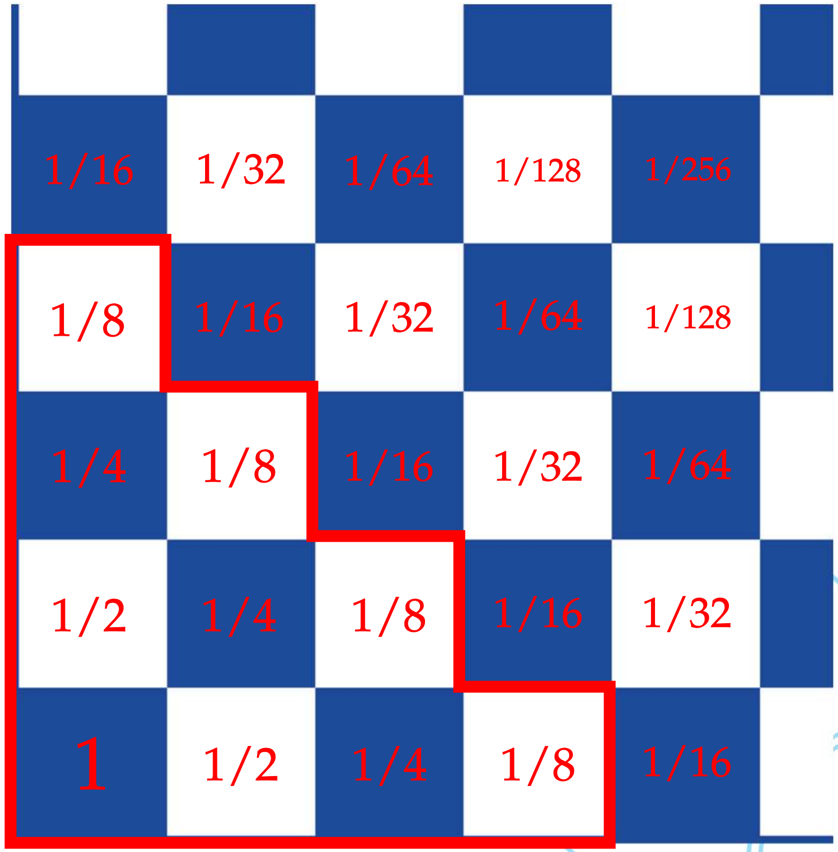
At each step, total weight on the board doesn't change.

But, total weight outside of the box is less than

$$\frac{5}{16} + \frac{6}{32} + \frac{7}{64} + \cdots$$

which is less than 1!!!

Thus, it is not possible to have zero pebbles in the red box. QED



How these solutions are usually written…

To each lattice point (x, y) , assign a weight of $\left(\frac{1}{2}\right)^{x+y}$. Then, change of weight after each move is

$$-\frac{1}{2^{x+y}}+\frac{1}{2^{x+1+y}}+\frac{1}{2^{x+y+1}}=0$$

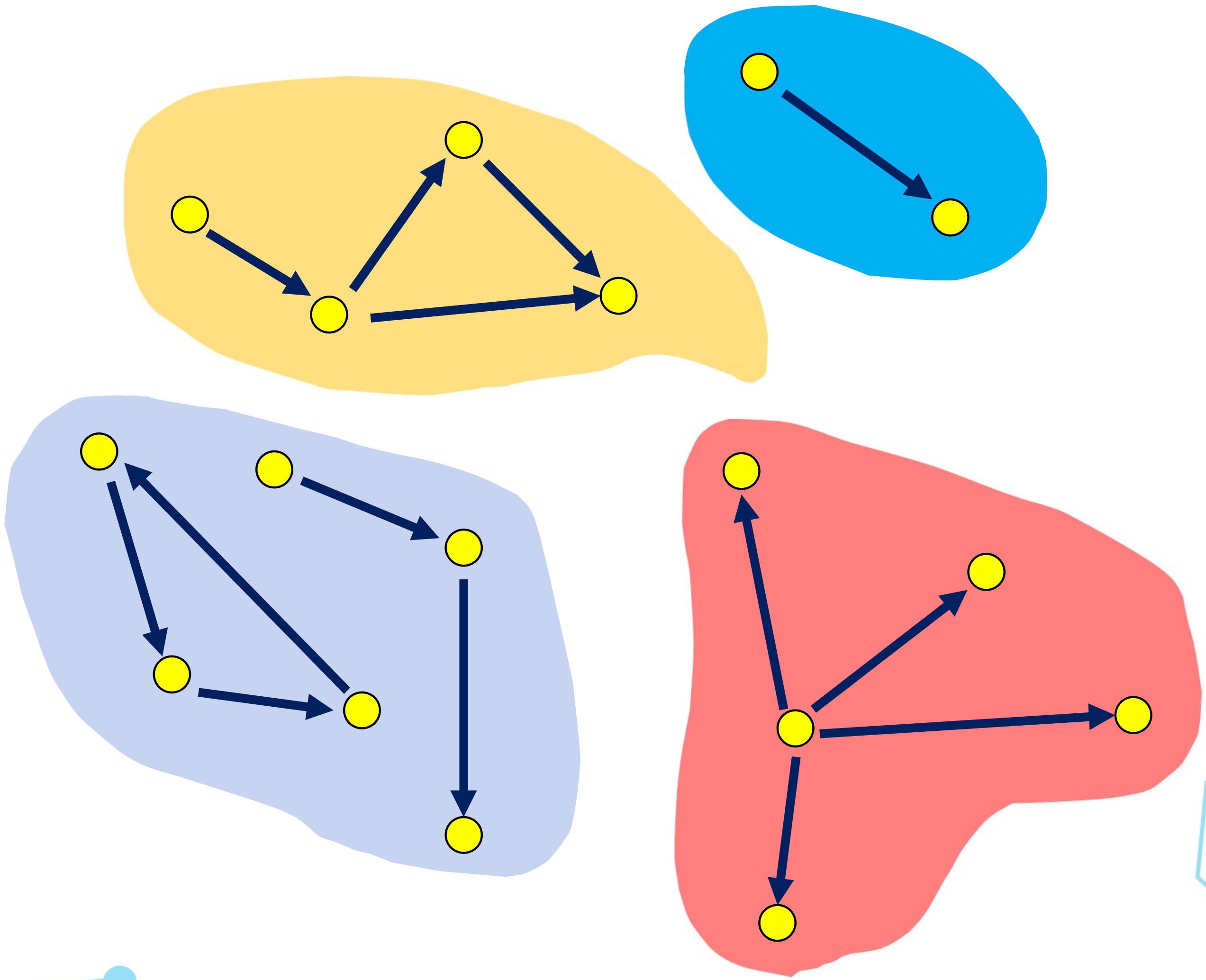
Thus, the total weight of lattice points that have pebbles on them is equal to 1 throughout the entire game. But, sum of all the weights above the line $x+y=3$ is

$$\sum_{k=4}^{\infty}\frac{k+1}{2^k}=\frac{3}{4}<1.$$

Therefore, there will always be a pebble on a lattice point either on the line or below the line $x+y=3$.

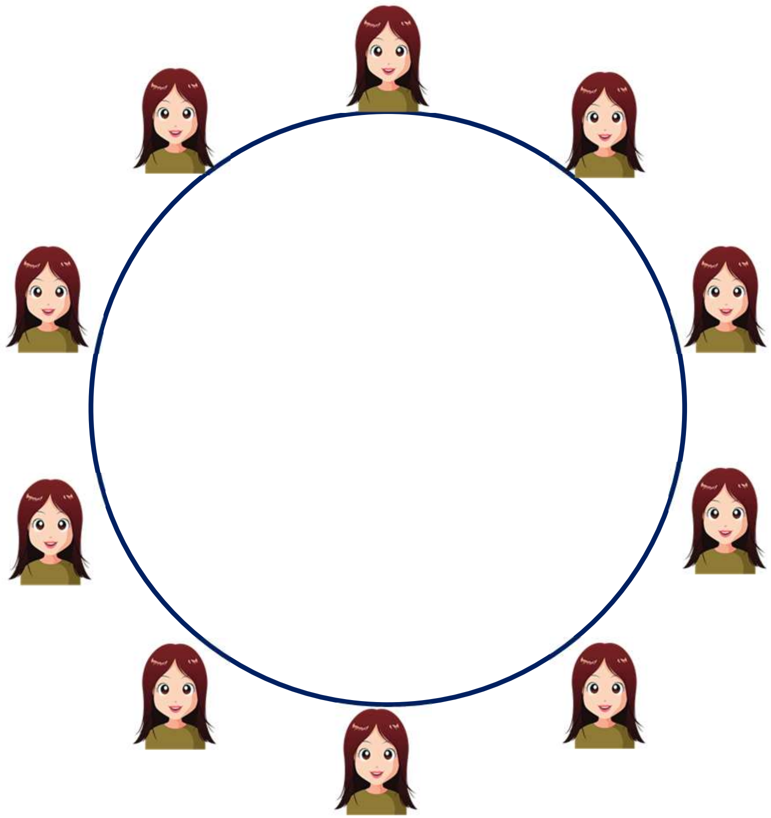
Remark: This problem is still true if we replace the line $x+y=3$ with $x+y=2$. Exercise :)

The Idea of Invariants

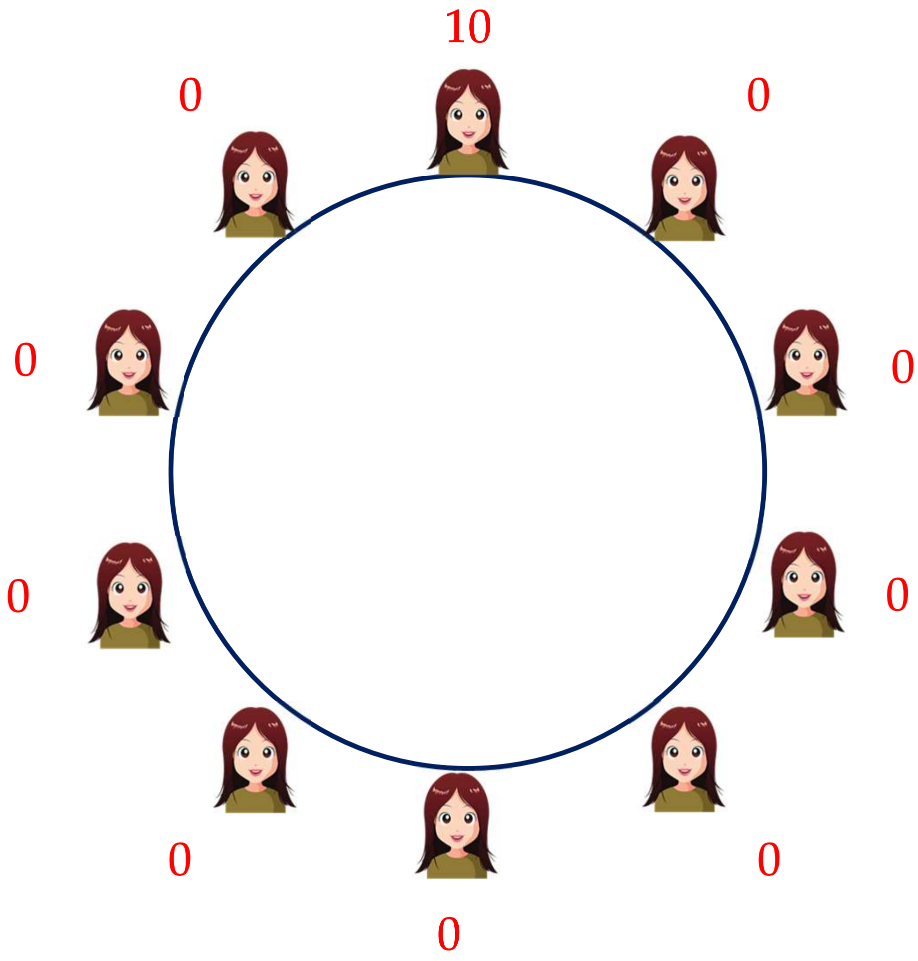


More Invariants: IMOSL1994/C5(b)

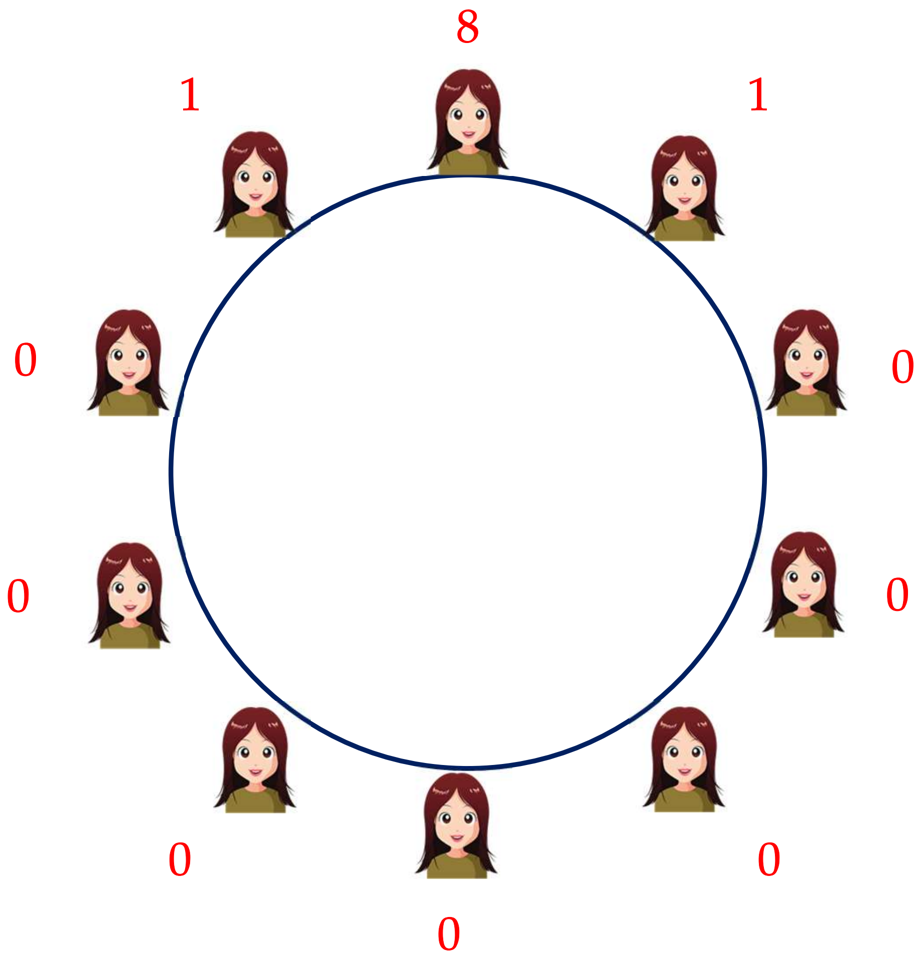
There are 1994 girls standing in a circle. Initially, one of the girls is holding 1994 coins. At every second, every girl that is holding at least two coins will give one coin each to her neighbors. Show that the game cannot terminate in finite number of seconds.



Example (with 10 girls & 10 coins)



Example (with 10 girls & 10 coins)



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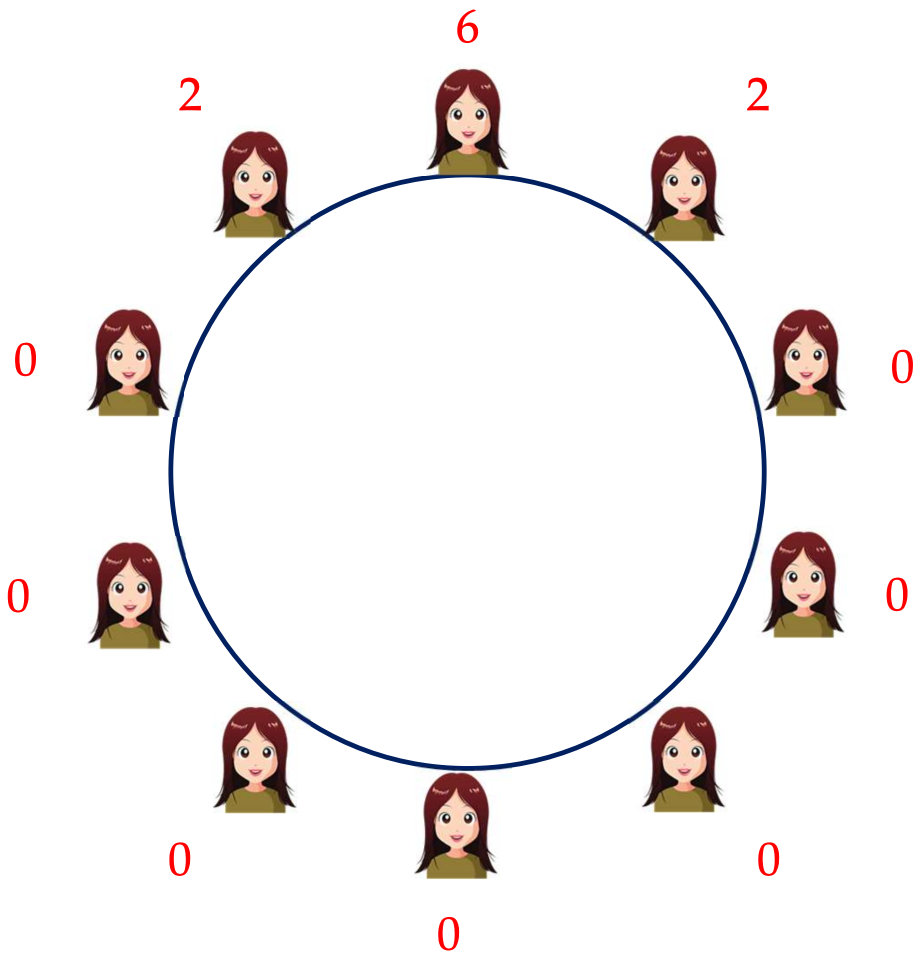

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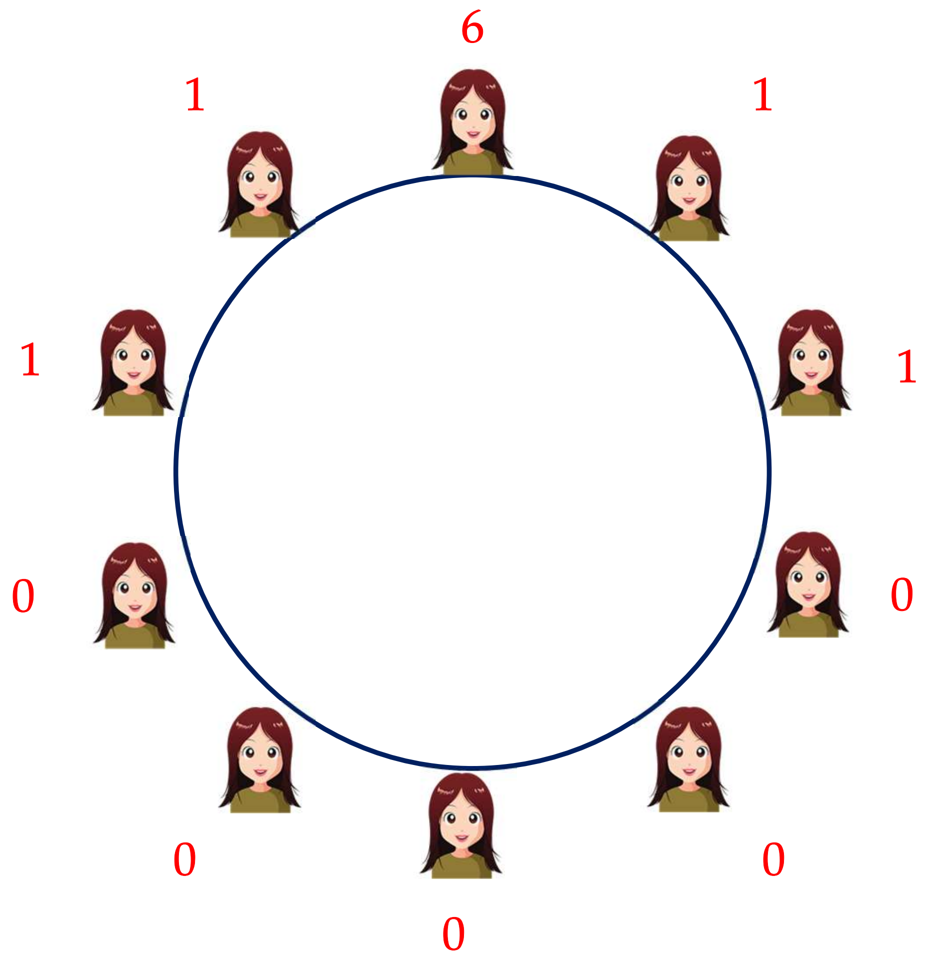

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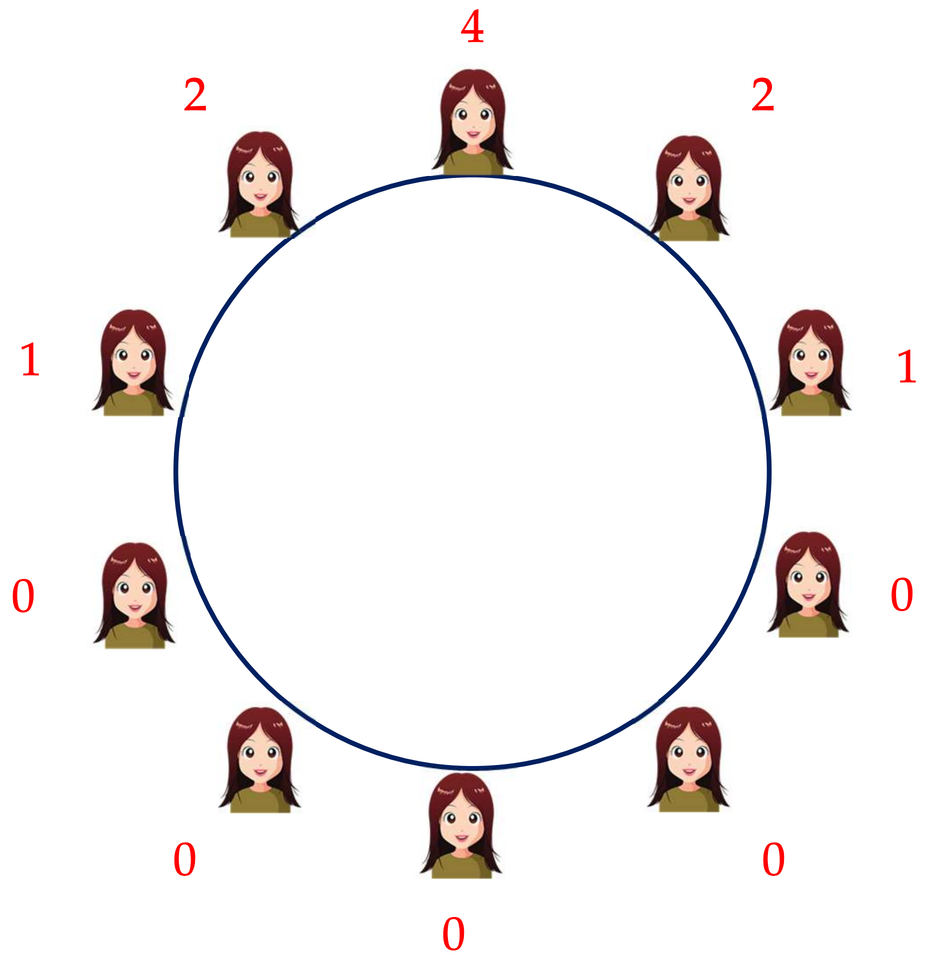
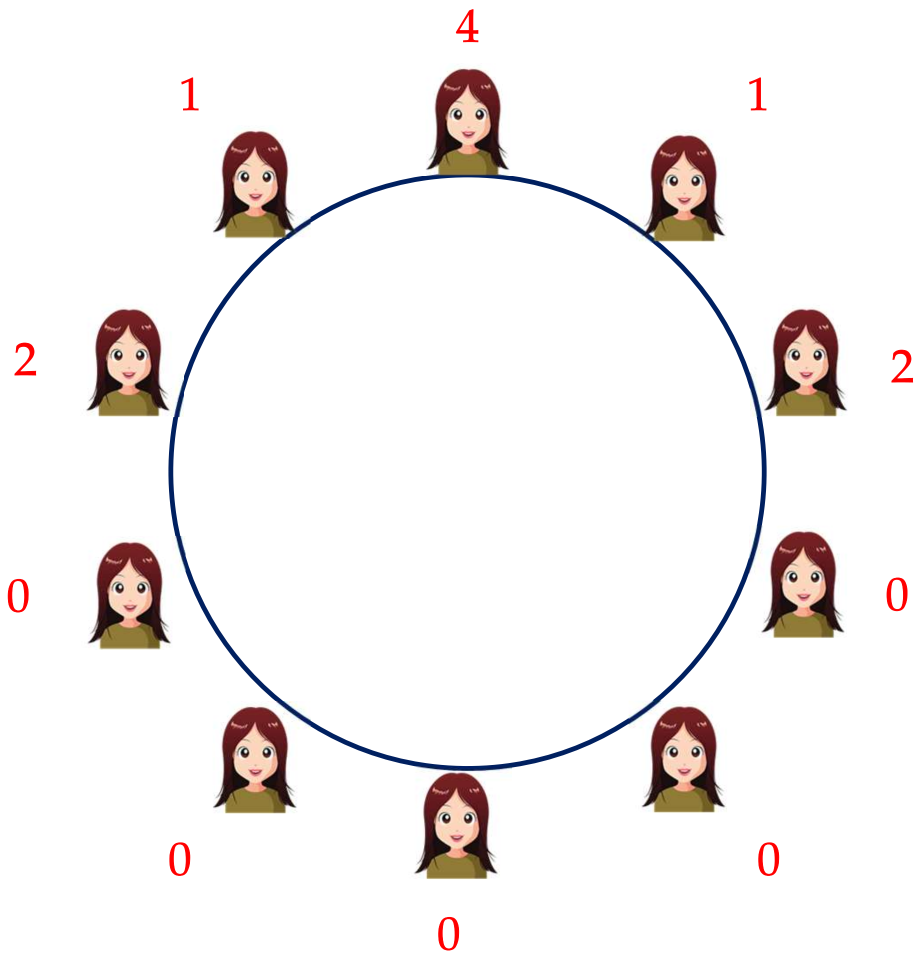


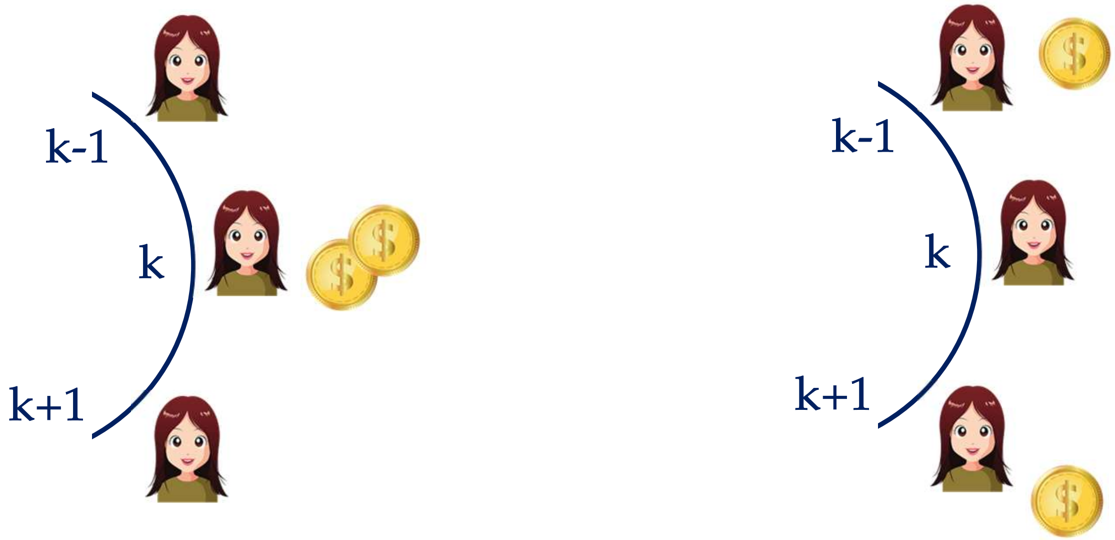
Example (with 10 girls & 10 coins)



Main Idea…

Label the girls $0, 1, 2, \ldots, 1993$ with 0 being the girl with all coins at the start.

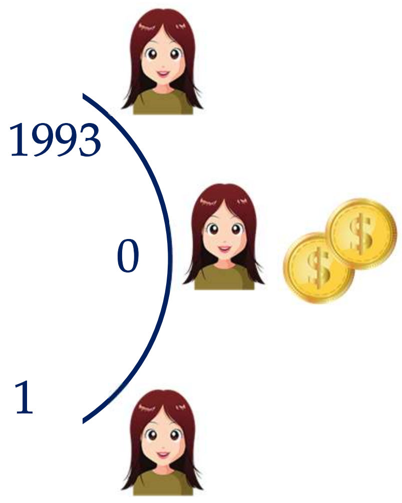
Think of the coins in the hands of girl k to be worth of k cents. Then, …



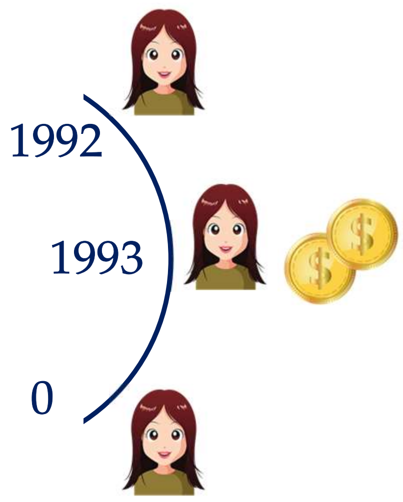
So… total worth of coins do not change???

The Edge Cases

This is not true if $k = 0$ or 1993. What happens in these cases?



Increases by 1994



Decreases by 1994

So, total weight does not change mod 1994.

Proof

First, we started with total worth of 0.

Whenever a girl passes her coins, total worth mod 1994 doesn't change.

Thus, total worth is always divisible by 1994.

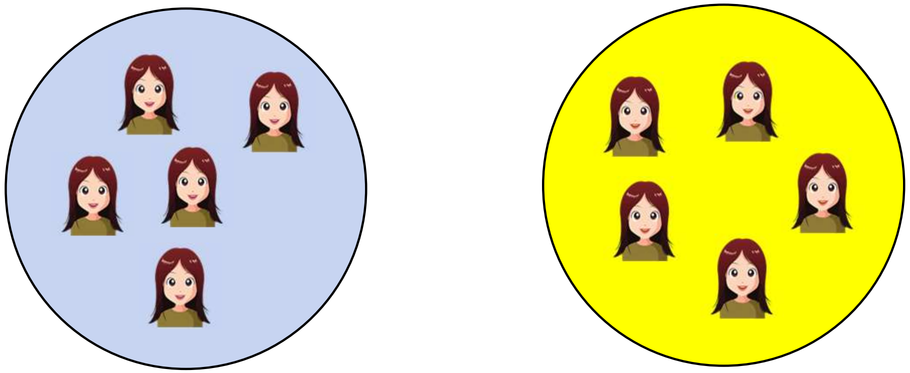
But, if the game were to end, everyone will have one coin each in last second. So, in this case, total worth will be

$$0 + 1 + 2 + \cdots + 1993 = \frac{1993 \times 1994}{2} = 1993 \times 997 \equiv -997 \pmod{1994}$$

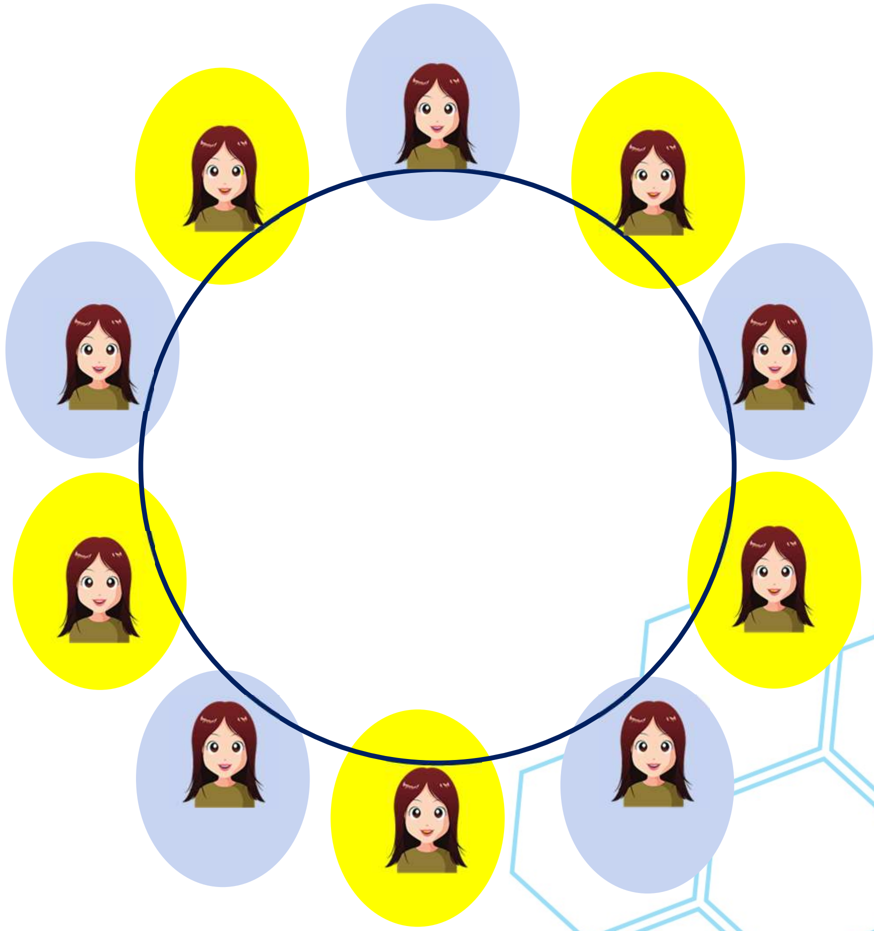
Hence, the game will never end!

Alternate Solution…

Divide the girls into two groups: yellow and cyan by putting odd numbered girls into yellow and the rest into cyan. Then,

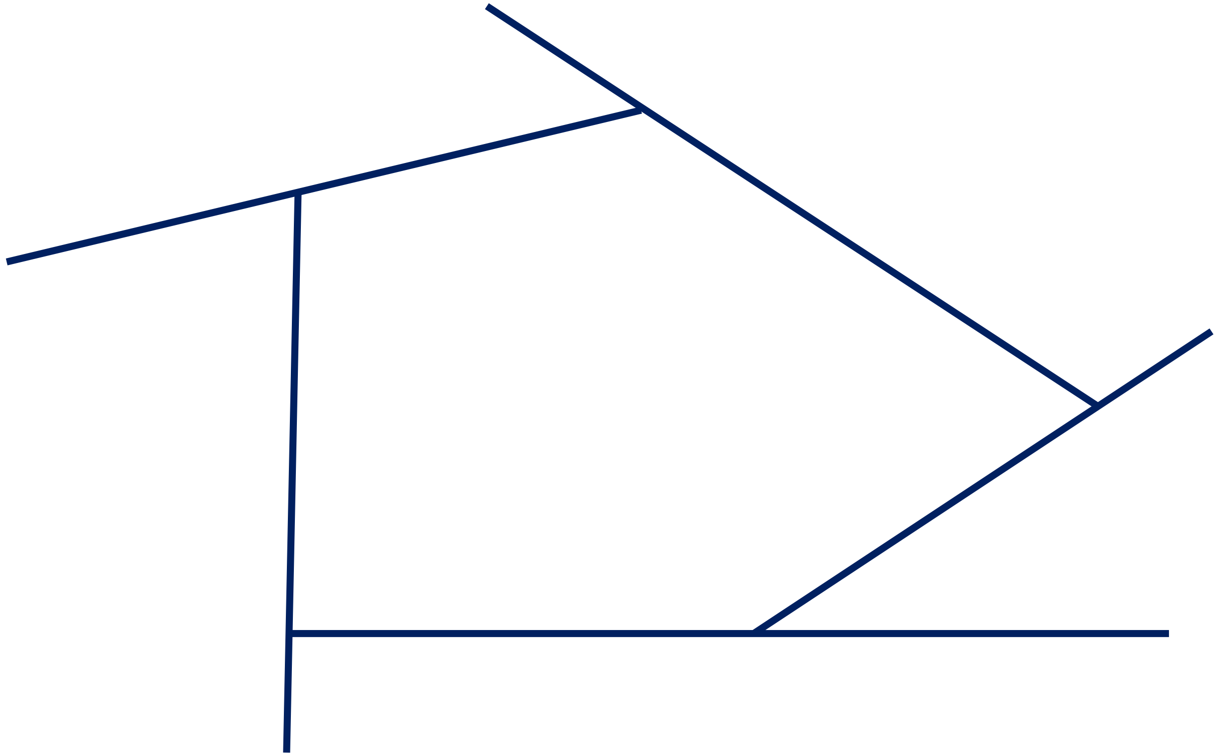


Number of coins in each group change by 2 for each pass. So, parity of the number of coins in each group does not change! Cyan group starts with even, but will have odd number of coins if the game were to end!



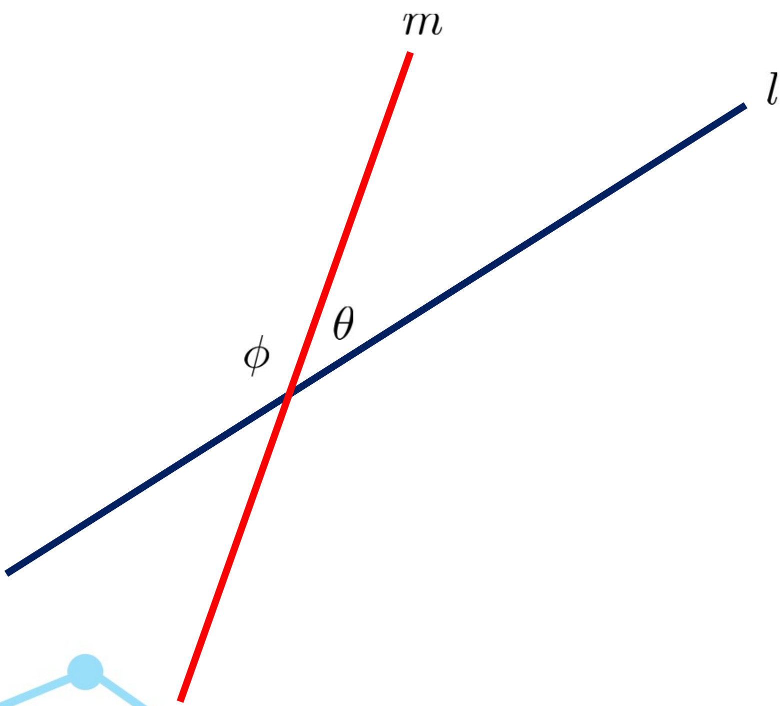
Let’s do some geometry…

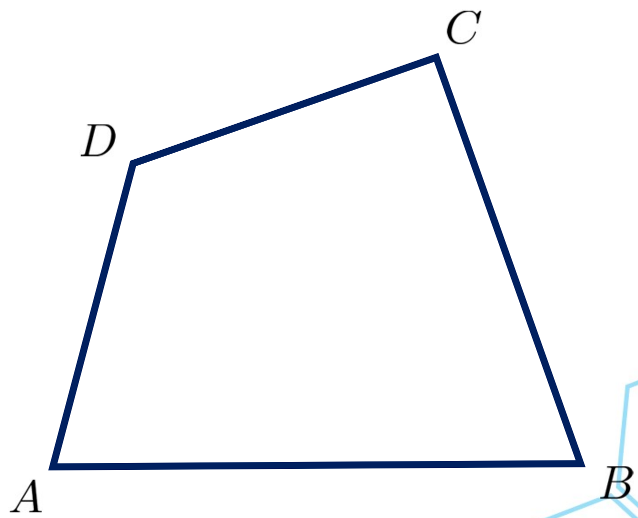
Show that the sum of exterior angles of a convex polygon is equal to 360-degrees.



Directed Angles

Given two lines l and m, write $\angle(l,m)$ for the counterclockwise angle that you have to rotate l to get a line parallel (or equal) to m. In the picture, $\angle(l,m)=\theta$, $\angle(m,l)=\phi$.



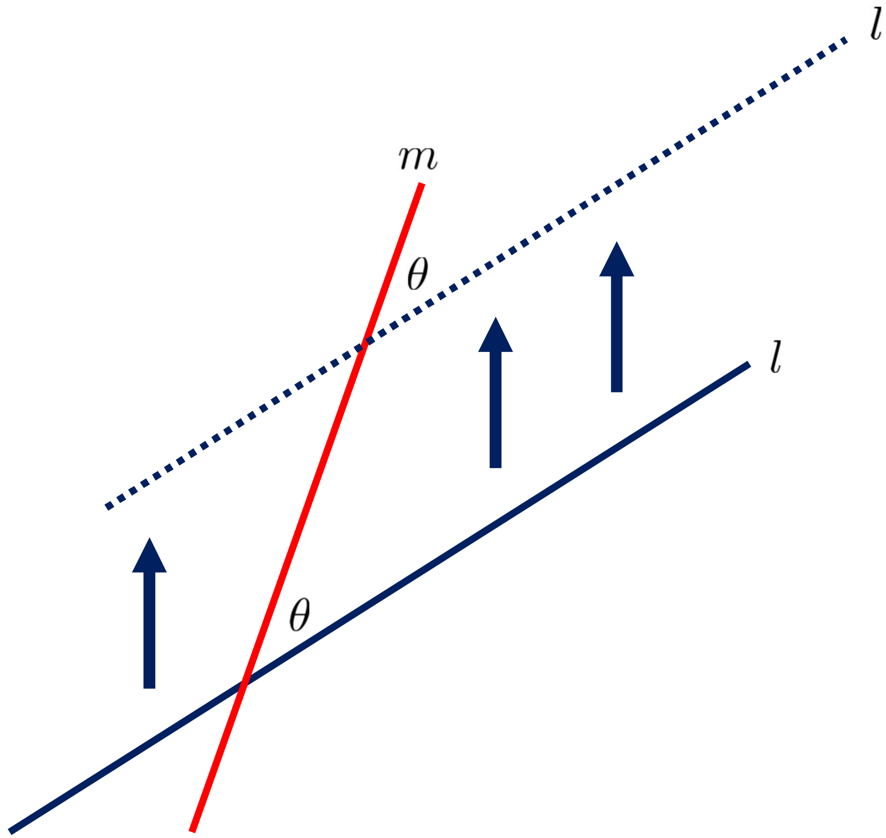


sum of exterior angles

$$= \angle(AB,BC)+\angle(BC,CD)+\angle(CD,DA)+\angle(DA,AB)$$

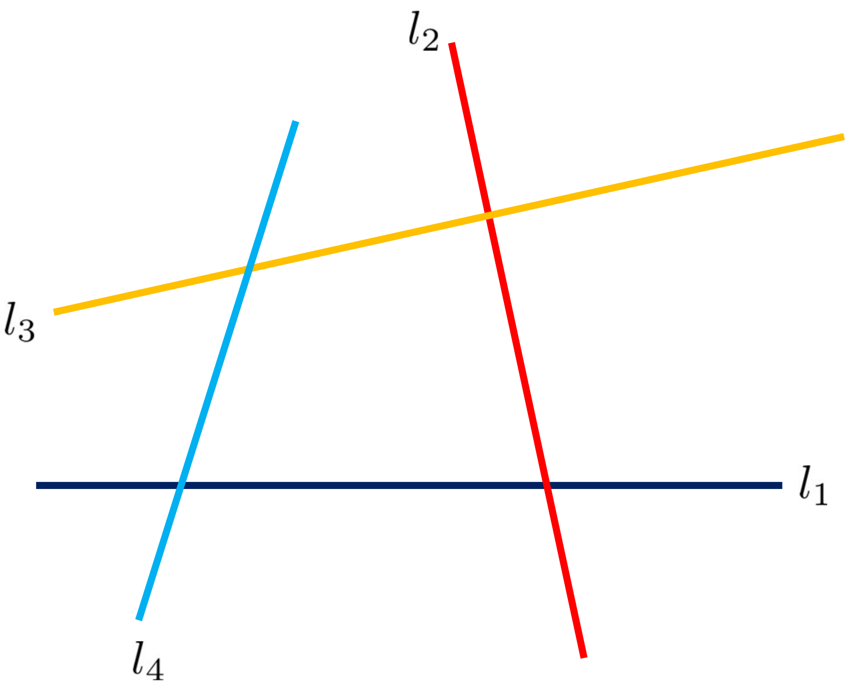
The Invariant

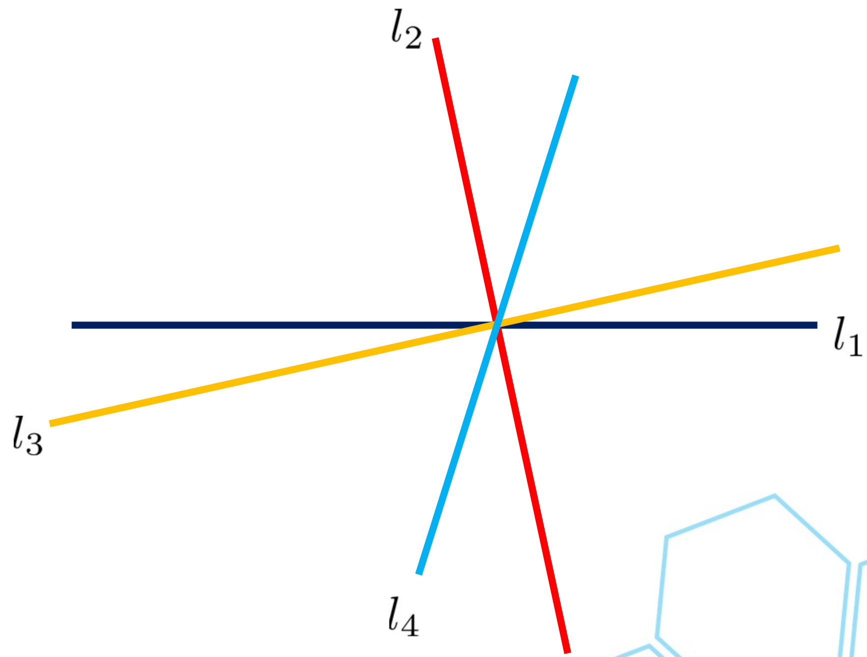
Fix a line m and "move" a line l. If the movement is via translation, then the directed angle between them does not change.



Proof

Simply translate the sides of the polygon so that all the sides become concurrent!





$$\angle(l_1, l_2) + \angle(l_2, l_3) + \angle(l_3, l_4) + \angle(l_4, l_1) = 360^\circ$$

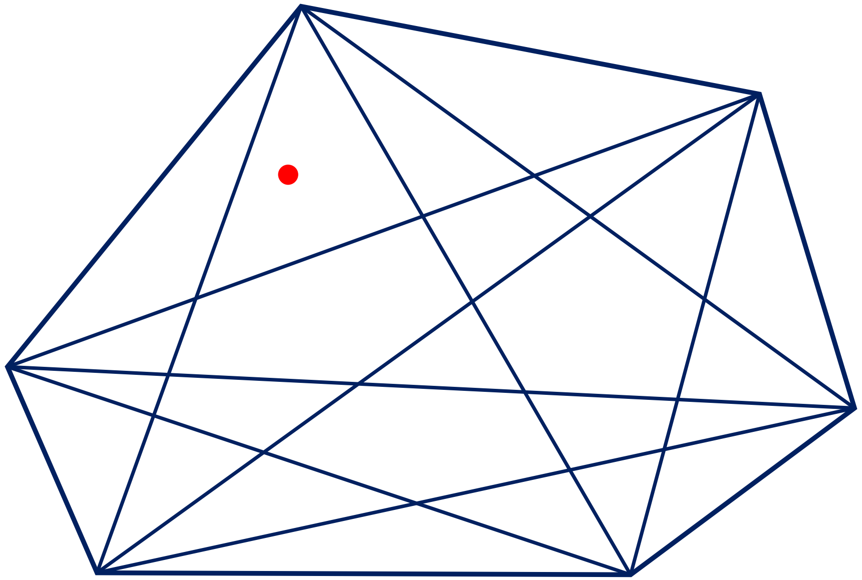
Warning: This proof still needs more rigor… For example, where did we use convexity???

Invariants can be used…

- To show that we cannot get from position A to position B using allowed moves,
- To show that the final configuration is unique,
- To carry a property from a simple position to a complicated position,
- …

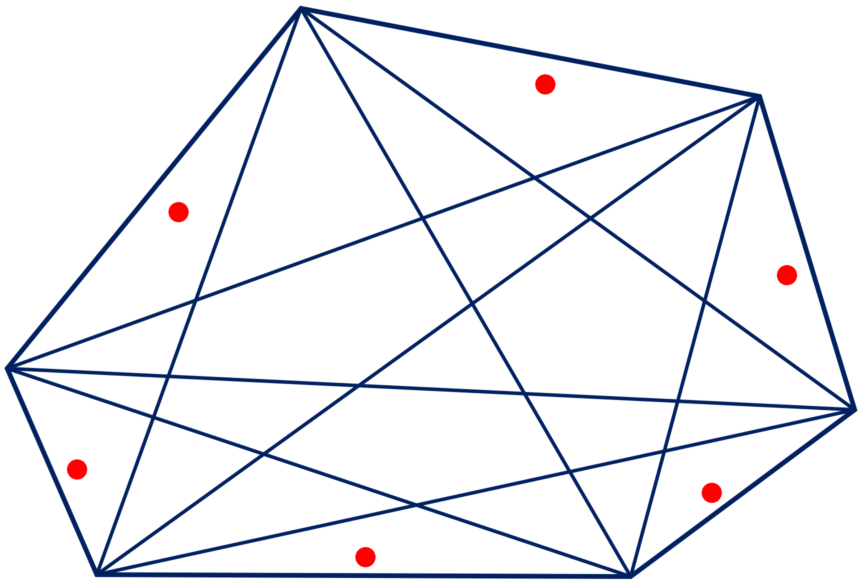
Property Carrying

Consider a convex $2n$-gon. Show that every point inside the polygon, not lying on the diagonals, lies inside an even number of triangles determined by the vertices of the polygon.



Example

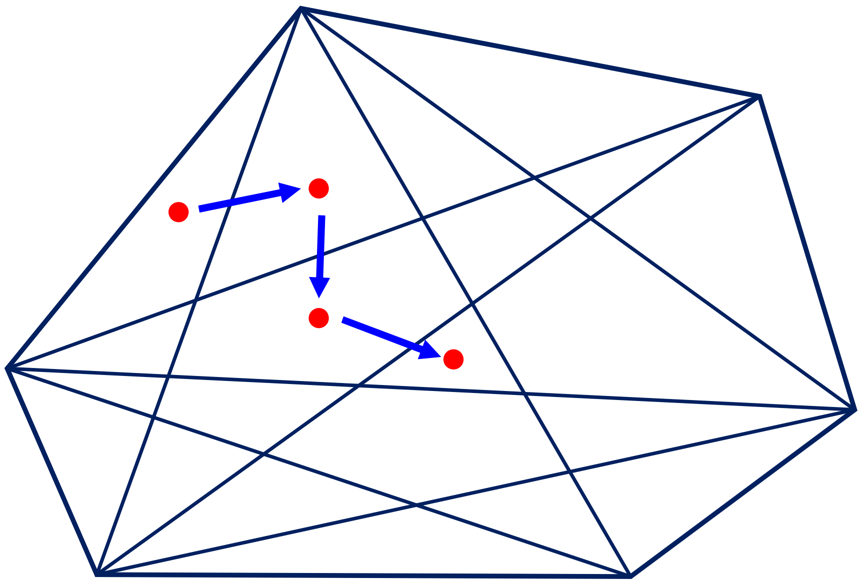
Thankfully, counting is easier near the edges…



Each red point lies inside $2n - 2$ triangles.

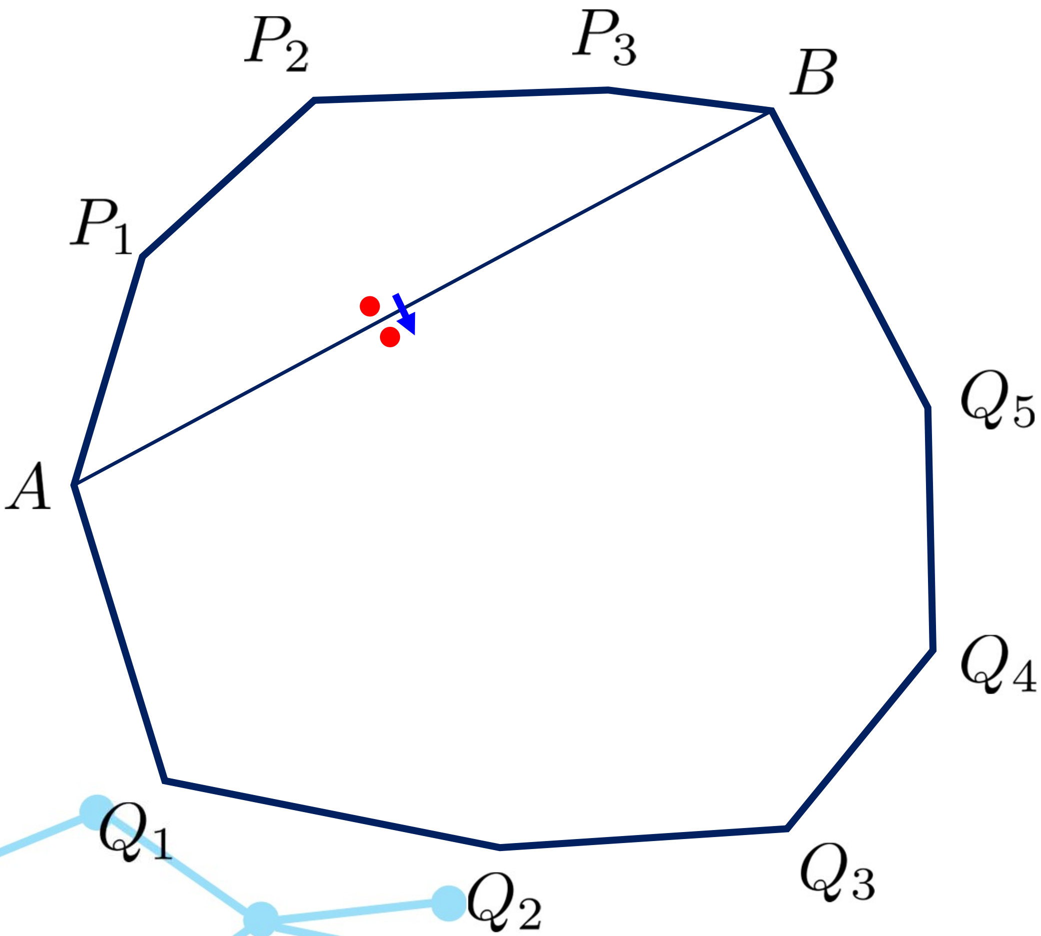
Main Idea

To Prove: If we move a point across the boundary, our property does not change!



Problem Solving Tip: Figuring out what to prove is as helpful as making observations.

Main Idea (Proof)



If we move across diagonal AB,

Triangles lost: ΔABP_i

Triangles gained: ΔABQ_j

Number of P's and number of Q's have same parity.

Thus, change in number of triangles is even.

Let’s have a break!

See you on Problem Solving Session!