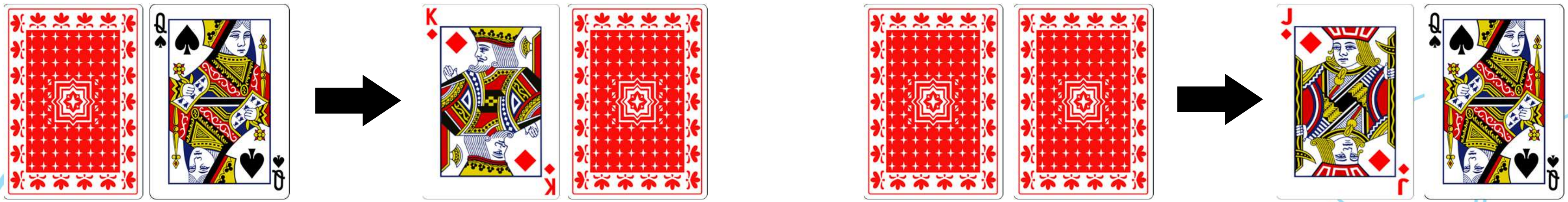
Lecture – 1

Monovariants

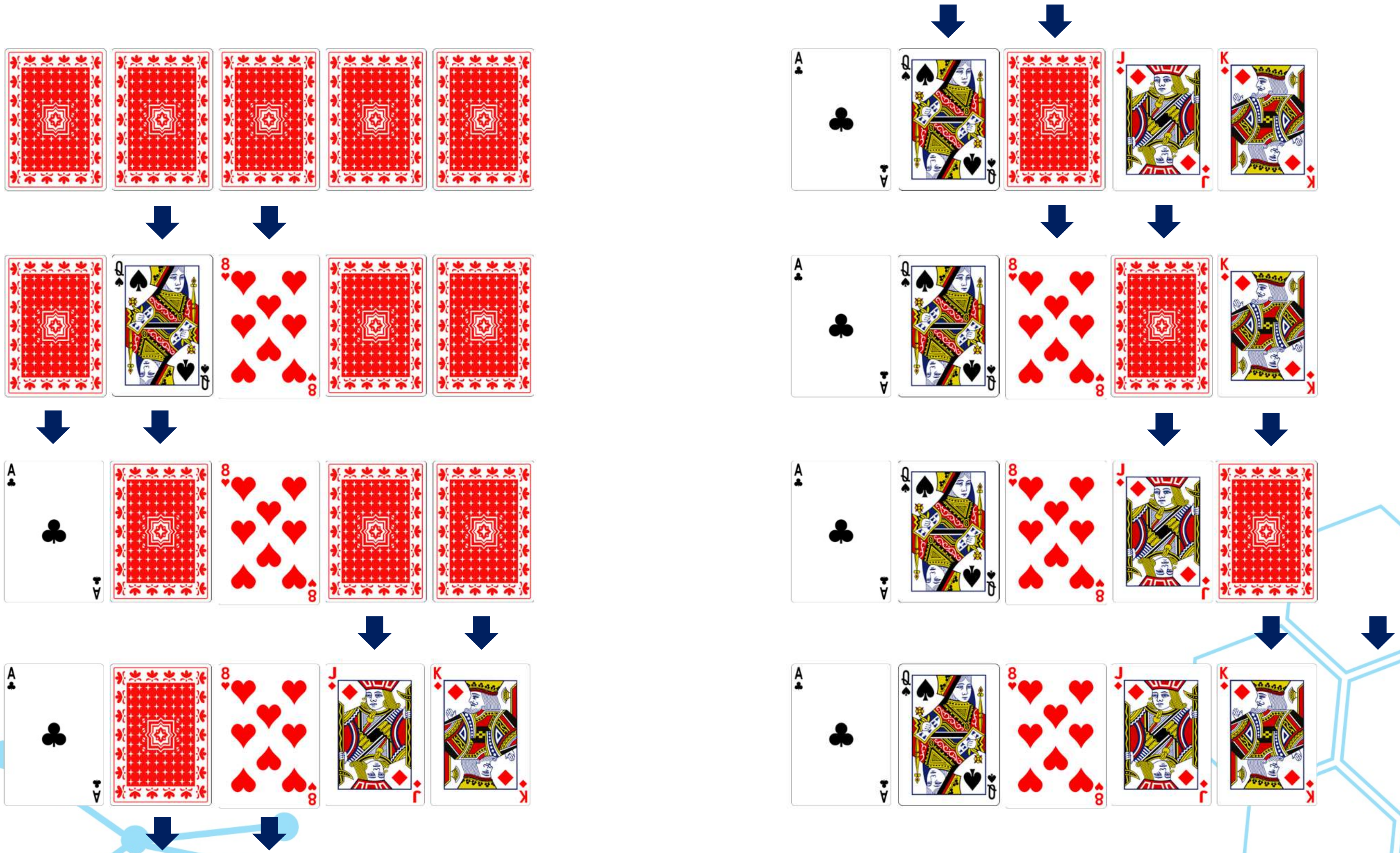
A Clip from $X + Y$

Problem Statement

Twenty random cards are placed in a row, all face-down. A **move** consists of turning a face-down card face-up and turning the card immediately to the right (if there is no card to the right, we only turn the face-down card). Show that no matter what the choice of cards to turn, this sequence of moves must terminate.

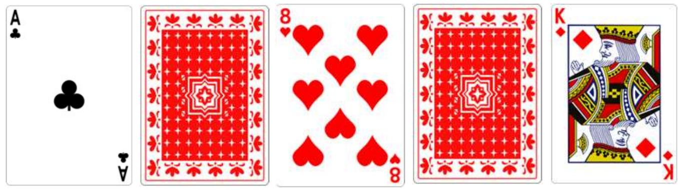


Example with 5 cards



Let’s Solve on Our Own First

Observation: Face-up cards on the very left can never be changed.



Bold Claim: Twenty may be replaced with any n, initial position doesn’t matter!

1. This means that if the left-most card is face-up, we are done! (why?)
2. Call this card C. If we ever flipped C in our process, then we are done.
3. If we did not flip C, then what happens?
4. Then by induction, all cards to the right of C will eventually be face-up.
5. So then, we will be forced to flip C, so we are done!

Writing up

We prove by induction that the conclusion holds if 20 is replaced by any positive integer n despite the initial configuration. When $n = 1$, the process obviously terminates. Now, suppose we have proved our assertion for all positive integers $1, 2, \dots, n-1$. We will now prove the assertion on n. Arrange n cards in a row and call the left-most card C. We have two cases:

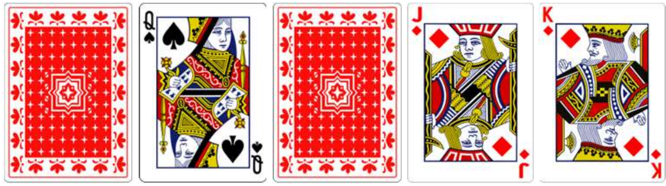
Case 1: Suppose C is face-up. Then, note that none of the moves can make C face-down. Therefore, all our moves are performed on the $n-1$ cards to the right of C and hence by inductive assumption, every card will be face-up and the process must terminate.

Case 2: Suppose C is face-down. If C is changed to face-up at any moment, then we are done by case 1. So, suppose (to the contrary) that C is never changed to face-up. But then, our moves are performed on the $n-1$ cards to the right of C and hence by inductive assumption, all cards except C will eventually be face-up. Then, we are forced to change C to face-up, contradiction.

A Clip from $X + Y$ continued

Nathan's Solution

1. Think of face-up as 0 and face-down as 1.



$\longleftrightarrow$ 10100

2. Then, our moves become: $10 \to 01$ and $11 \to 00$.

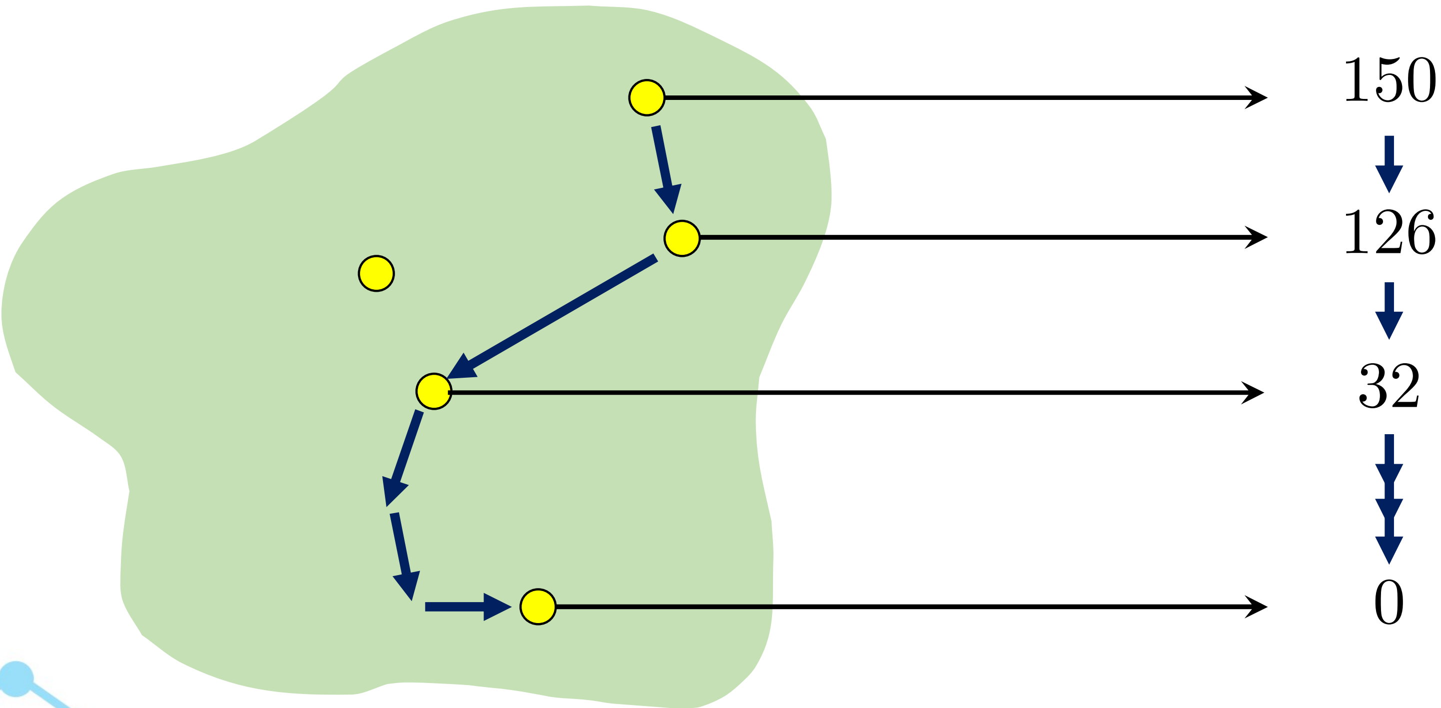
10100 $\rightarrow$ 01100 $\rightarrow$ 00000

3. No matter which move we do, the string of 0s and 1s, seen as a number, will decrease.

4. Therefore, the process must terminate!

Idea of Monovariants

To each "configuration" X, assign a number $f(X)$ (or anything that can be ordered). Show that under the "allowed moves", $f(X)$ always decreases or increases (by some constant).



Idea of Monovariants

Be careful! The increment/decrement must be “large”; for example, by some constant. Otherwise, the situation like the following may occur!

$$0.1 \longrightarrow 0.01 \longrightarrow 0.001 \longrightarrow 0.0001 \longrightarrow \cdots$$

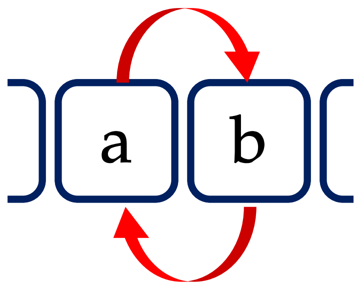
But, if we work in $\mathbb{Z}_{\geq 0}$, this situation will never happen.

What Monovariants are Capable of…

- Proving that a particular process eventually terminates.
- Bounding the number of steps before termination.
- Comparing the positions in some sense.
- …

Sorting Problem

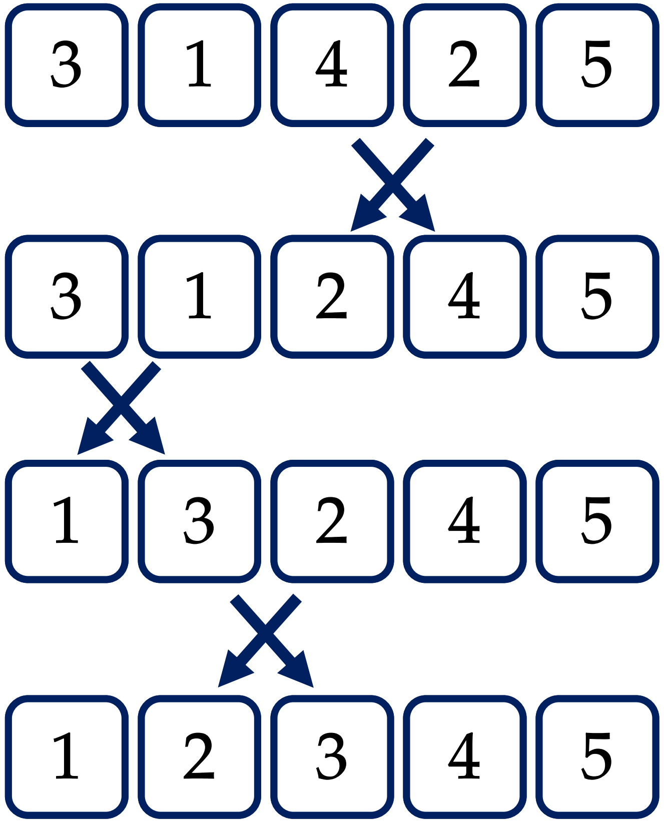
Timothy finds n cards laid in a row on his table, bearing numbers 1 through n with one number on each card. He wants to sort the cards so that the numbers on the cards are increasing from left to right. To do so, he iteratively picks two consecutive cards with a written on the left card and b on the right such that $a > b$, and then swaps them.



1. Show that he will eventually sort the cards eventually.

2. What is the maximum number of swaps does he need?

Example



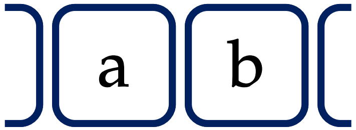
Let’s Try to Find a Monovariant

3	1	4	2	5

- What happens if we think of this as a 5-digit number?
- Yes! The number decreases with each step, so this is indeed a monovariant.
- So, the process eventually stops! But how can we determine the maximum number of swaps???

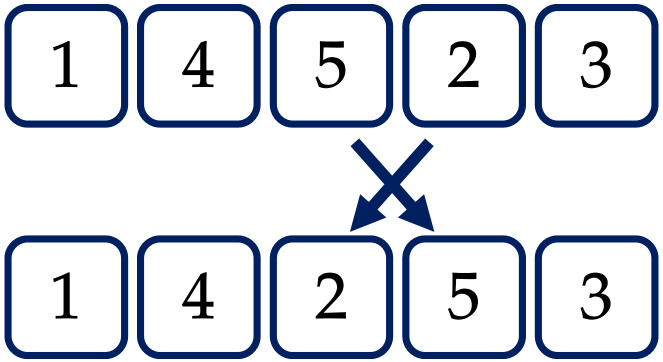
This is very difficult to do with the current monovariant.

Let’s Try to Find Another Monovariant



Call a pair of consecutive cards **inverted** if the card on the left bears a larger number.

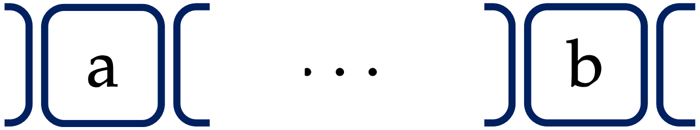
Does the number of **inverted** consecutive pairs decrease with each swap?



1 inverted pair

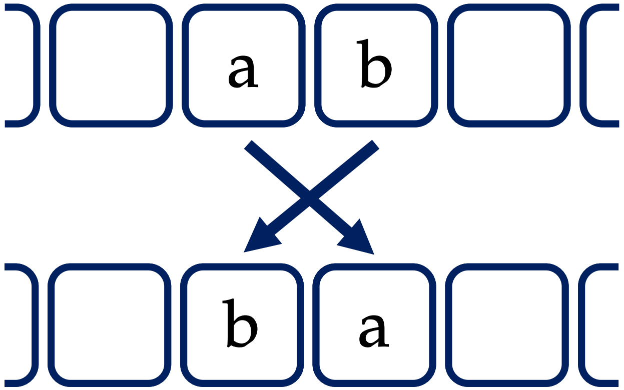
2 inverted pairs!

Let's Try to Find Another Monovariant



Call a pair of ~~consecutive~~ cards **inverted** if the card on the left bears a larger number.

Does the number of inverted pairs decrease with each swap?



Doesn't change!

After each swap,

- # inversions containing a and others
- # inversions containing b and others
- # inversions between the others
- # inversions between a and b

Decreases by 1

Let’s Try to Bound!

Now, the argument becomes easy…

- With each swap, number of inversions decreases by 1.
- Number of inversions is $\leq n(n-1)/2$
- Thus, number of swaps is $\leq n(n-1)/2$ as well!

Is there a configuration that needs exactly $n(n-1)/2$ swaps? YES!
If cards were initially in descending order, every pair is inverted.
Thus, we will need exactly $n(n-1)/2$ swaps to remove them all.

So, maximum number of swaps needed is $n(n-1)/2$.

Writing up is homework… :(

What is USATSTST?

Let me just explain the entire USA IMO team selection process instead!

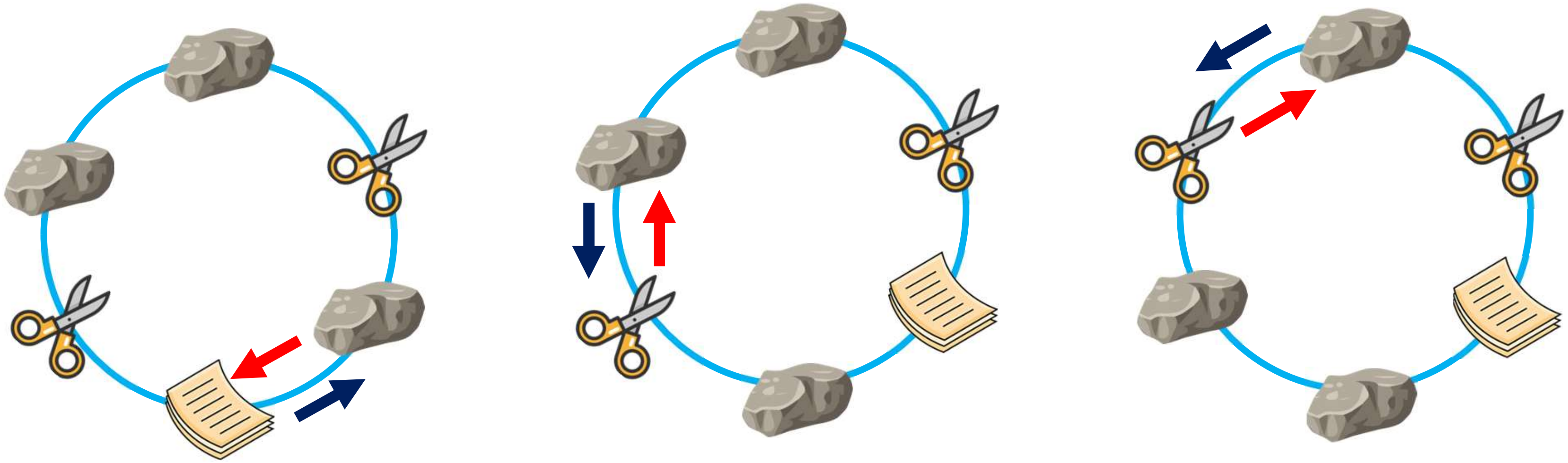
USATSTST 2020 Problem 1

Let a, b, c be fixed positive integers. There are $a + b + c$ ducks sitting in a circle, one behind the other. Each duck picks either rock, paper or scissors, with a ducks picking rock, b ducks picking paper and c ducks picking scissors. A **move** consists of an operation of one of the following three forms:

- If a duck picking rock sits behind a duck picking scissors, they switch places.
- If a duck picking paper sits behind a duck picking rock, they switch places.
- If a duck picking scissors sits behind a duck picking paper, they switch places.

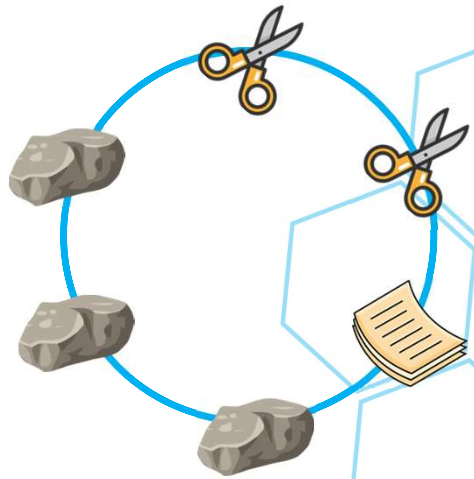
Determine, in terms of a, b and c, the maximum number of moves which could take place, over all possible initial configurations.

Example



Observation:

We cannot do any more moves if and only if the configuration looks like the one on the right…



What do we have to do…?

What does the following equality mean?

the maximum number of moves that could take place over all possible initial configurations is equal to M

It means that **both** of the following are true…

- There is a configuration in which M moves take place. ← Usually easier
- Every configuration comes to stop within M moves.

Three Things We Need…

Our brain needs to do three jobs to finish this problem…

- Make an educated guess of M. ⟵ Most Important!
- Show that "there is a configuration in which M moves take place."
- Show that "every configuration comes to stop within M moves."

Only these are necessary for writing the solution

The Solution Sketch…

We claim that the answer is M.

Part I : Consider the following initial position:

[describe a special position]
[make M moves in that position]

… Therefore, in this position, M moves may take place.

Shows that M is achievable

Part II : Now, suppose we have an arbitrary starting position.

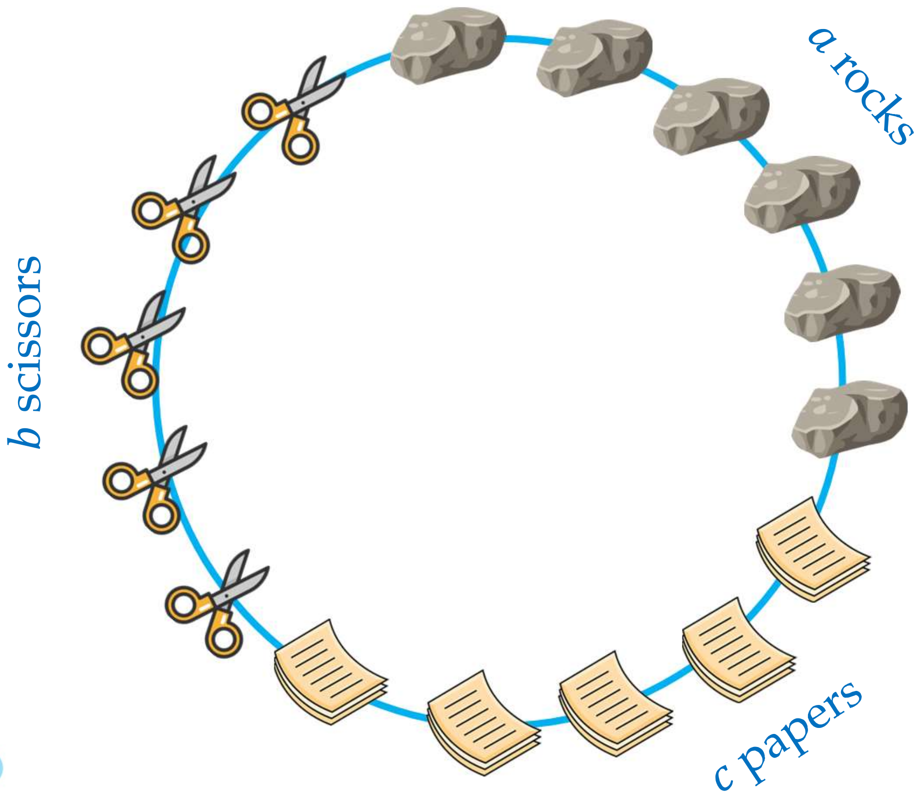
[make some creative argument]

Therefore, we won't be able to make more that M moves.

Shows that we cannot get better than M

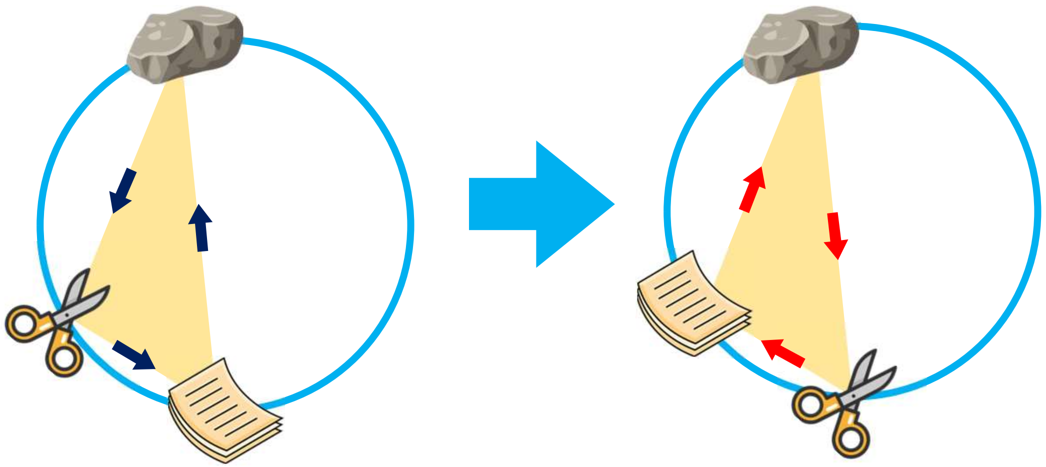
Where were we…

Guessing M is hard… So, let's try to come up with an initial position in which you can make a LOT of moves…



Let's Find a Monovariant…

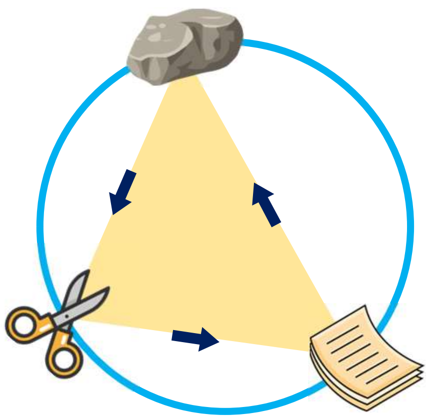
What happens when we swap scissors and paper?



Triangle  changes orientation!

Let's Find a Monovariant…

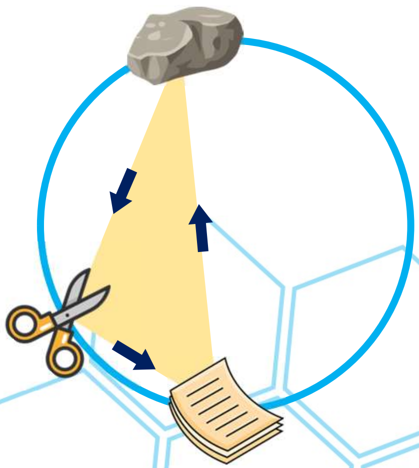
So, let's consider the number of positively oriented  triangles!



If we swap scissors and paper, this number decreases by a (one for each rock).

paper and rock, decreases by b (one for each scissors).

rock and scissors, decreases by c (one for each paper).



So, we have found a monovariant! Better, we can know how much it changes with our moves!

Let’s Try to Bound…

We started with __________ positively oriented 🪨✂️📄 triangles.

Each swap decreases this number by a, b or c.

If $a \leq b \leq c$, then number of moves is at most $\dfrac{abc}{a} = bc$.

In general, the number of moves is at most $\dfrac{abc}{\min\{a, b, c\}} = \max\{ab, bc, ca\}$.

Writing up

We claim that the answer is $\max\{ab, bc, ca\}$. First, consider the initial configuration where a rock-ducks stay consecutive, b scissors-ducks stay consecutive and c paper-ducks stay consecutive in clockwise order. Then, by only making moves between two of the three types of ducks, we can see that $\max\{ab, bc, ca\}$ moves can be made.

Now, suppose we start with an arbitrary initial configuration. Call a triplet of one rock-duck, scissors-duck and paper-duck around the circle in clockwise order, a good triple. Note that we initially start with at most abc good triples. Whenever we swap scissors-duck and paper-duck this number decreases by a. Similarly, this number decreases by b or c in other cases. Therefore, the number of moves we can make is at most

$$\frac{abc}{\min\{a, b, c\}} = \max\{ab, bc, ca\}$$

What Monovariants are Capable of…

- Proving that a particular process eventually terminates.
- Bounding the number of steps before termination.
- Comparing the positions in some sense.
- …

That’s it for today!

See you on ______ .