Video – 4

Stars&Bars Trick and Combinatorial Identities

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Section – I

Stars and Bars

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Q1. Marbles Problem Review

Ko Thway has an infinite supply of red, blue, yellow and black marbles. Marbles of the same colour are indistinguishable. He wants to make a collection of 6 marbles. How many different collections can he create?

Q1. Marbles Problem Review

Solution

The collections can be represented by permutations of 6 stars and 3 bars.

For example,

2 red, 2 blue, 1 yellow, 1 black: ★ ★ | ★ ★ | ★ | ★

1 red, 5 blue, 0 yellow, 0 black: ★ | ★ ★ ★ ★ ★ | |

2 red, 0 blue, 2 yellow, 2 black: ★ ★ | | ★ ★ | ★ ★

Therefore, the answer is $\dfrac{9!}{6! \times 3!} = 84.$

Stars and Bars Trick

Suppose we want to select n objects from the given k types. Suppose also that we have infinite (or sufficient) supply of objects for each type.

Then, the selections can be represented with a permutation of n stars and $k-1$ bars.

Therefore, number of ways is $\dfrac{(n+k-1)!}{n! \times (k-1)!}$.

Note: The final answer is not that useful. Just remember the "representing selections using permutations" trick.

Q2. Stars and Bars Practice

Followings can be represented as permutations of some stars and some bars. How may stars and how many bars?

- Selections of 10 marbles from a infinite supply of red, blue, green, pink and teal. ⟵ 10 stars, 4 bars
- Ways to distribute 20 identical apples to 6 children. ⟵ 20 stars, 5 bars
- Selections of 7 letters from English alphabet where the same letter can repeat. ⟵ 7 stars, 25 bars
- Non-negative integer solutions to $x + y + z = 30$. ⟵ 30 stars, 2 bars

Q3. Pencil Distribution

Soe Soe wants to distribute 10 identical pencils to 5 of her students. She wants it so that every student gets at least one pencil. In how many ways can she distribute?

Solution

Given one pencil to each student first.

Now, we have to distribute 5 pencils to 5 students without worrying about the restriction.

So, the distributions can be represented as 5 stars and 4 bars.

Thus, number of ways is $\dfrac{9!}{5! \times 4!} = 126.$

Q4. Restricted Integer Solutions

How many positive integer solutions are there to the equation $x + y + z = 12$ so that $x \leq 4$?

Solution

Put 1 into x, y and z first.

So, we just need to find number of non-negative integer

solutions to $x + y + z = 9$ with $x \leq 3$.

Number of solutions with $x > 3$ is $\dfrac{7!}{5! \times 2!} = 21$.

Number of solutions with $x \geq 0$ is $\dfrac{11!}{9! \times 2!} = 55$.

Therefore, number of solutions with $x \leq 3$ is $55 - 21 = 34$.

Section – II

Combinatorial Proofs

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$1 + 2 + 3 + \cdots + n$ by counting

Question: Ten people wants to play a round-robin tournament. How many matches will be played?

Answer: Number of ways to select 2 people out of 10 is $C_2^{10} = \dfrac{9 \times 10}{2}$.

Another answer: First person plays 9 matches. Second person has to play 8 more matches. Third person has to play 7 more matches, ... , ninth person has to play 1 more match. Thus, answer is $1 + 2 + 3 + \cdots + 9$.

This proves that $1 + 2 + 3 + \cdots + 9 = \dfrac{9 \times 10}{2}$.

$1 + 2 + 3 + \cdots + n$ by counting

Question: $n + 1$ people wants to play a round-robin tournament. How many matches will be played?

Answer: Number of ways to select 2 people out of n is $C_2^{n+1} = \dfrac{n(n+1)}{2}$.

Another answer: First person plays n matches. Second person has to play $n - 1$ more matches. Third person has to play $n - 2$ more matches, ... Thus, answer is $1 + 2 + 3 + \cdots + n$.

This proves that $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

Q5. $C_r^n = C_{n-r}^n$

Let $0 \leq r \leq n$ be positive integers. Prove that $C_r^n = C_{n-r}^n$.

Solution

(Note: Suppose $r \geq 1$)

Let's count the number of ways to select r people out of n to form a team.

By definition, number of ways = LHS.

But, selecting r people to be included is the same as selecting $n - r$ people to be excluded.

Therefore, number of ways = RHS.

Hence, LHS = RHS.

Note: For $r = 0$, both LHS and RHS evaluates to 1.

Pascal's Triangle

Try to write down the values of C_r^n in a triangular grid.

$$\begin{array}{cccccccccccc} & & & & & C_0^0 & & & & & \\ & & & & C_0^1 & & C_1^1 & & & & \\ & & & C_0^2 & & C_1^2 & & C_2^2 & & & \\ & & C_0^3 & & C_1^3 & & C_2^3 & & C_3^3 & & \\ & C_0^4 & & C_1^4 & & C_2^4 & & C_3^4 & & C_4^4 & \\ C_0^5 & & C_1^5 & & C_2^5 & & C_3^5 & & C_4^5 & & C_5^5 \end{array} \longrightarrow \begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

What do you notice?

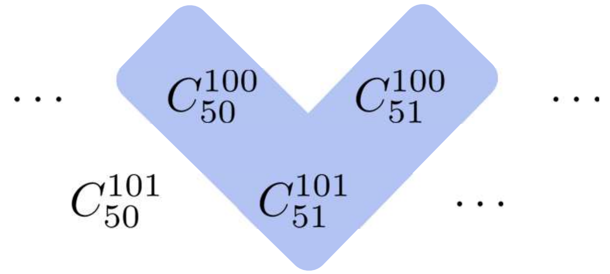
Q6. $C_r^n + C_{r+1}^n = C_{r+1}^{n+1}$.

Let $0 \leq r < n$ be positive integers. Prove that $C_r^n + C_{r+1}^n = C_{r+1}^{n+1}$.

Solution

(Note: Suppose $r \geq 1$. Proving for $r = 0$ is left as exercise)

Let's count the number of ways to select $r + 1$ apples out of $n + 1$ to buy.



By definition, number of ways = RHS.

Now, choose your favourite apple, call it A.

Number of ways that include A is equal to C_r^n.

Number of ways that does not include A is equal to C_{r+1}^n.

So, total number of ways = $C_r^n + C_{r+1}^n$ = LHS.

Q7. Simplifying

Simplify $C_5^5 + C_5^6 + C_5^7 + \cdots + C_5^{2023}$.

Solution

Note that $C_5^5 = C_6^6$

So,

$$C_6^6 + C_5^6 = C_6^7$$

$$C_6^7 + C_5^7 = C_6^8$$

$$\cdots$$

$$C_6^{2023} + C_5^{2023} = C_6^{2024}$$

Therefore, $C_5^5 + C_5^6 + C_5^7 + \cdots + C_5^{2023} = C_6^{2024}$.

That's it for this video.

See you soon!

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