Video – 2

Fun Puzzles in Combinatorics

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Section – I

Truths and Lies

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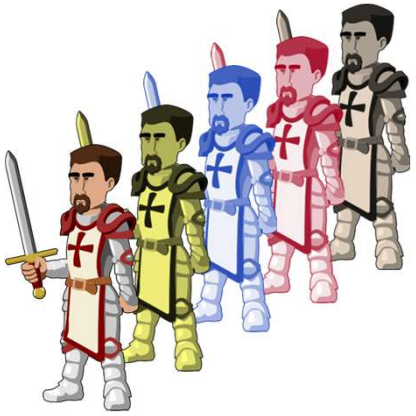
Q1. Knights in a Queue

On an island, only two types of knights live: one type only tells truth, and the other only tells lies. One day, 2023 knights were found queuing in a line, and all of them except the one at the front of the queue say



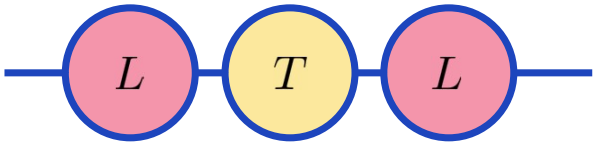
"the knight in front of me is a liar."

How many liars are there in the queue? There is more than one possibility.

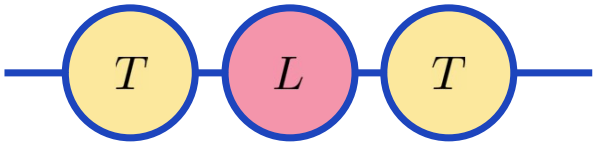
Q1. Knights in a Queue

Solution

- What do we know if a knight T is a truth teller?
  - The person in front of T is a liar.
  - The person behind T is also a liar.



- What do we know if a knight L is a liar?
  - The person in front of L is a truth-teller.
  - The person behind L is also a truth-teller.



Therefore, truth-tellers and liars alternate in the queue.

So, number of liars is either 1011 or 1012.

Q2. Kids Lie on Halloween (AMC10)

On Halloween 31 children walked into the principal's office asking for candy. They can be classified into three types: Some always lie; some always tell the truth; and some alternately lie and tell the truth. The alternaters arbitrarily choose their first response, either a lie or the truth, but each subsequent statement has the opposite truth value from its predecessor. The principal asked everyone the same three questions in this order.

- "Are you a truth-teller?" The principal gave a piece of candy to each of the 22 children who answered yes.
- "Are you an alternater?" The principal gave a piece of candy to each of the 15 children who answered yes.
- "Are you a liar?" The principal gave a piece of candy to each of the 9 children who answered yes.

How many pieces of candy in all did the principal give to the children who always tell the truth?

Q2. Kids Lie on Halloween (AMC10)

Solution There are four types of kids:

- Type T: Always tells truth, ⟵ Answers: Yes, No, No
- Type L: Always tells lie, ⟵ Answers: Yes, Yes, No
- Type AT: Alternates and starts with truth, ⟵ Answers: No, No, No
- Type AL: Alternates and starts with lie. ⟵ Answers: Yes, Yes, Yes

Last column tells us that there are 9 ALs.

Middle column tells us that there are $15 - 9 = 6$ Ls.

First column tells us that there are $22 - 6 - 9 = 7$ Ts.

Since each T gets one candy, the principal gave 7 candies to T-students.

Section – II

Double Counting Tricks

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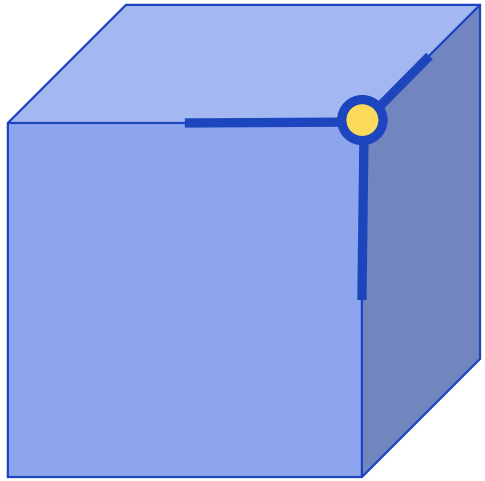
Q3. Numbers on a Cube (OMMO)

Ko Thiha writes a number on each vertex of a cube. Then, on each edge, he writes the sum of the numbers at its ends. Finally, on each face, he writes the sum of the numbers at its boundary edges. It turns out that the numbers written on the faces are 1, 2, 3, 4, 5, 6 as in a regular die. Find the sum of the numbers written on the vertices.

Q3. Numbers on a Cube (OMMO)

Solution

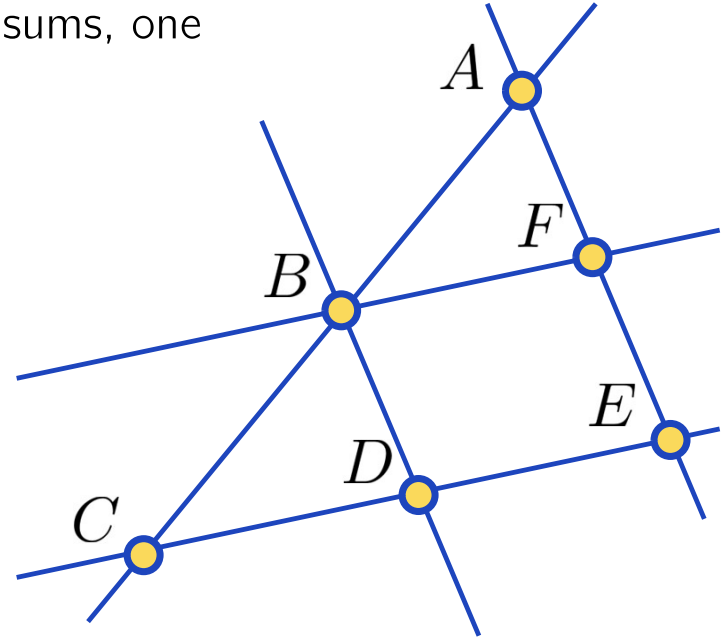
- Let sum of the numbers on vertices be S.
- Then, sum of the numbers on edges is $3S$.
- Then, sum of the numbers on faces is $2 \times 3S = 6S$.
- Since $6S = 1 + 2 + 3 + 4 + 5 + 6$, we have $S = 3.5$.



Q4. Lines and Sums (AMC8)

Each of the points A, B, C, D, E and F in the figure represents a different digit from 1 to 6. Each of the five lines shown passes through some of those points. The digits along each line are added to create five sums, one for each line. The total of those sums is equal to 47.

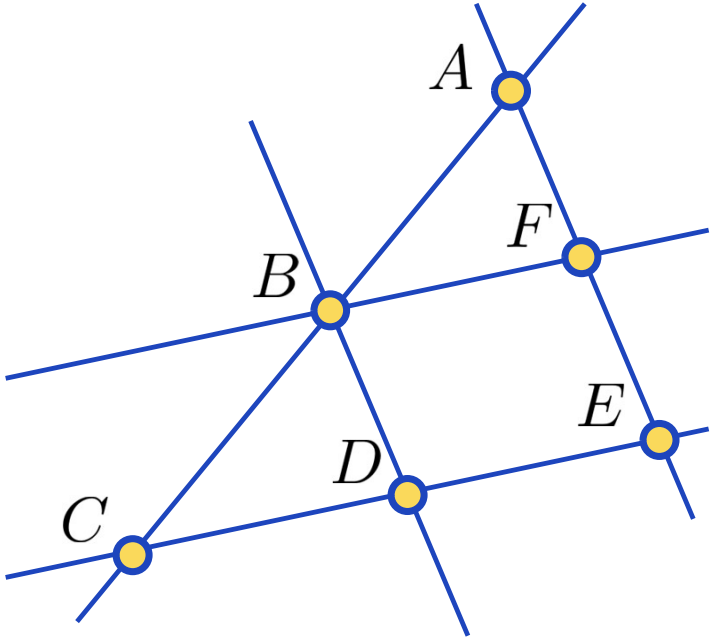
What is the digit represented by B?



Q4. Lines and Sums (AMC8)

Solution

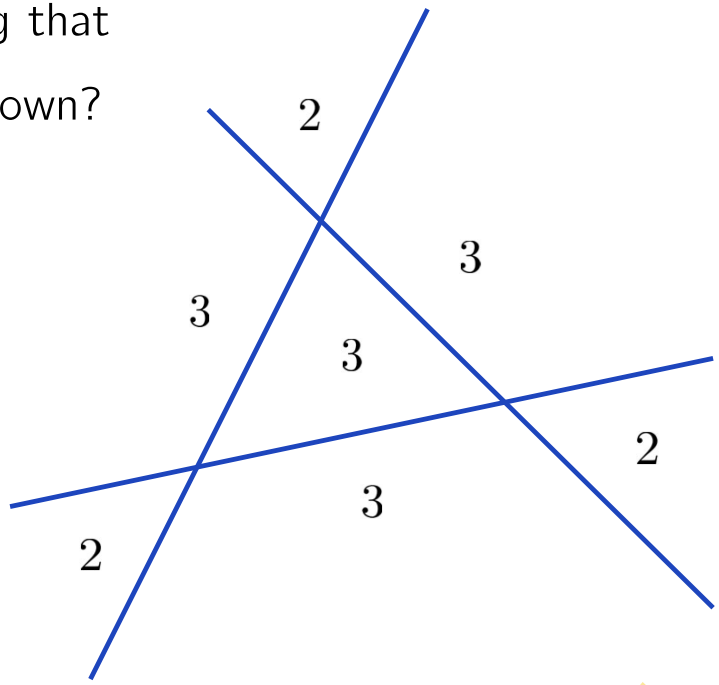
- 47 is equal to sum of $2A$'s, $3B$'s, $2C$'s, $2D$'s, $2E$'s and $2F$'s.
- So, 47 is equal to B plus two times $A + B + C + D + E + F$.
- So, 47 is equal to B plus 42.
- Hence, B is 5.



Question: How many possible values of A are there?

Q5. Regions and Sums (OMMO)

Twenty lines are drawn on the plane so that no two lines are parallel and no three lines are concurrent. These lines divide the plane into several regions. In each region, the number of line segments/rays bounding that region is written down. What the sum of all the numbers written down?



Q5. Regions and Sums (OMMO)

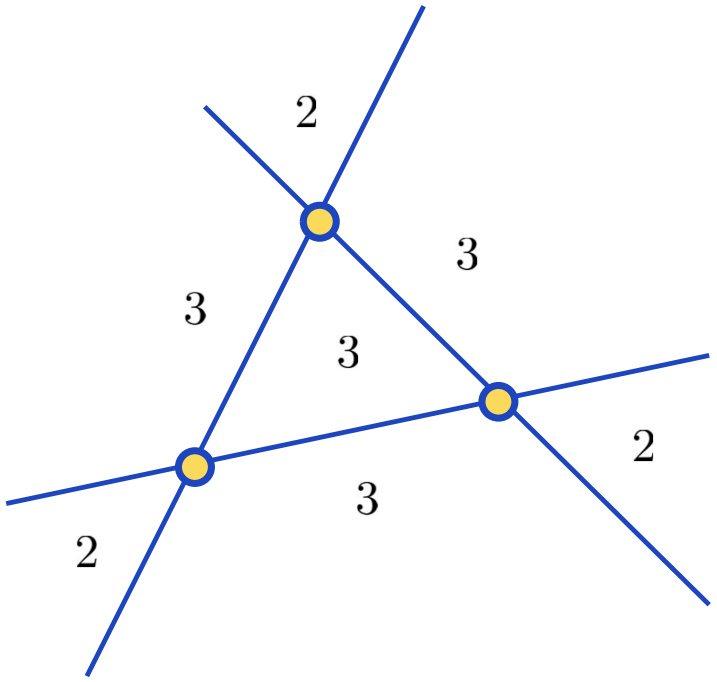
Solution

Main Observation: Sum of all the numbers is equal to twice the number of "parts" of the lines.

Each line is cut by 19 other lines, thus creating 20 such parts.

Thus, total number of parts is $20 \times 20 = 400$.

So, sum of all the numbers is $2 \times 400 = 800$.



Section – III

Counting Tiles

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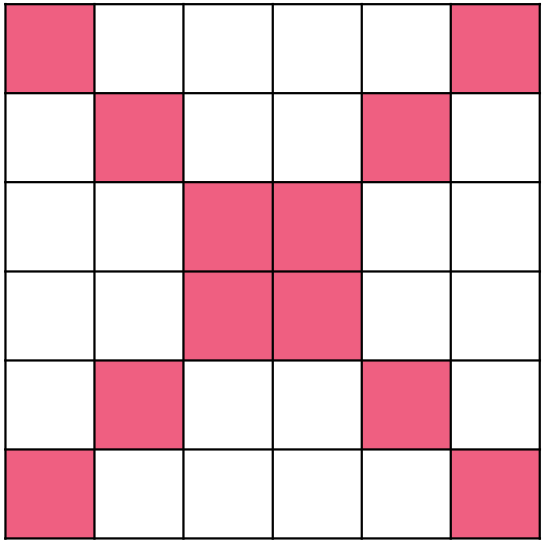
Q6. A Kitchen Floor Tiling (AMC8)

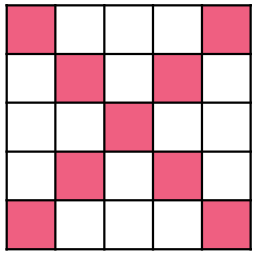
A square shaped kitchen floor is tiled with congruent square shaped tiles. A total of 37 tiles are used for the longest diagonals. How many tiles are used in total for the kitchen floor?

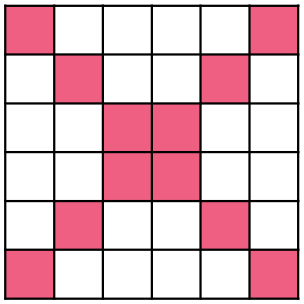


Q6. A Kitchen Floor Tiling (AMC8)

Solution

The "floor pattern" is different depending on the parity (even or odd) of the side-length.





Number of squares on diagonals is even if the side length is even.

Number of squares on diagonals is odd if the side length is odd.

Since 37 is odd, we must have the odd-pattern.

There are $(37 + 1)/2 = 19$ tiles on each side. So, there are 361 tiles in total.

Q7. P, Q and R (AMC8)

Letters P, Q and R are entered into the 20×20 grid in the following pattern.

How many P's, Q's and R's are there?

⋮	⋮	⋮	⋮	⋮	
Q	R	P	Q	R	...
P	Q	R	P	Q	...
R	P	Q	R	P	...
Q	R	P	Q	R	...
P	Q	R	P	Q	...

Q7. P, Q and R (AMC8)

Straightforward Solution

There are 3 types of columns:

- Type 1: PQR columns ⟵ 7P's, 7Q's, 6R's
- Type 2: QRP columns ⟵ 6P's, 7Q's, 7R's
- Type 3: RPQ columns ⟵ 7P's, 6Q's, 7R's

⋮	⋮	⋮	⋮	⋮	
Q	R	P	Q	R	...
P	Q	R	P	Q	...
R	P	Q	R	P	...
Q	R	P	Q	R	...
P	Q	R	P	Q	...

There are 7 Type 1s, 6 Type 2s and 6 Type 3s.

Therefore,

- Number of P's $= 7 \times 7 + 7 \times 6 + 6 \times 7 = 133$
- Number of Q's $= 7 \times 7 + 7 \times 7 + 6 \times 6 = 134$
- Number of R's $= 7 \times 6 + 7 \times 7 + 6 \times 7 = 133$

Q7. P, Q and R (AMC8)

More Elegant (and faster) Solution

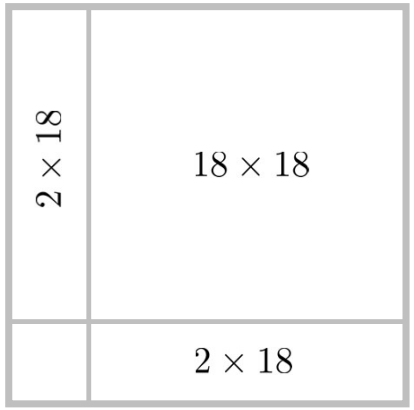
Main Observation: Note that number of P's, Q's and R's are the same in any 1×3 or 3×1 rectangle (call them trominoes).

If we remove the lower left 2×2 square, the resulting grid can be broken down into trominoes.

Therefore, there is 1 more Q than P's and R's.

Number of Q's $= 2 + \dfrac{400 - 4}{3} = 134$

⋮	⋮	⋮	⋮	⋮	
Q	R	P	Q	R	...
P	Q	R	P	Q	...
R	P	Q	R	P	...
Q	R	P	Q	R	...
P	Q	R	P	Q	...



Summary of this Video

We have used the following tips and tricks today:

- Make a small table: In the case where there are a lot of possible combinations, a table can help you think clearly.

- Try to "combine" everything: In some cases, you cannot know the values of individual unknowns. But, you can find something if you "combine all of them".

- Count the more convenient objects: Some objects are tedious (or maybe very difficult to count). But, we can sometimes write them in terms of easy-to-count objects.

That's it for this video.

Have a nice rest!

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