Video – 1

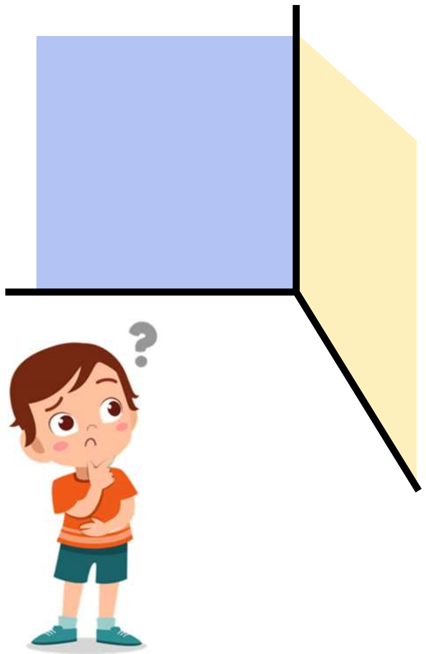
Further Tricks and Challenging Problems

Section – I

Counting by Cases

Q1: Painting a Room

Myint Thu has six different colours of paint available. He wants to paint the four walls of his box-shaped room using these colours so that adjacent walls receive different colours. In how many ways can he do this?

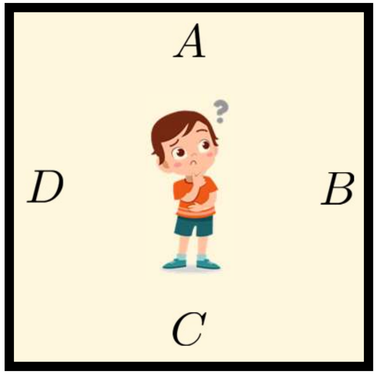


Answer: 630

Q1: Painting a Room

How NOT to do this question

Wall A	Wall B	Wall C	Wall D
6 choices	5 choices	5 choices	??? choices



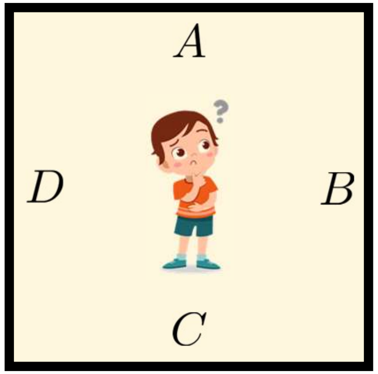
4 choices if A and C are different,
5 choices if A and C are same.

Q1: Painting a Room

Solution:

Group the colourings into two types.

- Type 1: A and C have different colours,
- Type 2: A and C have the same colour.



Counting Type 1

- Wall A: 6 choices
- Wall B: 5 choices
- Wall C: 4 choices
- Wall D: 4 choices

Number of type 1 $= 6 \times 5 \times 4 \times 4$

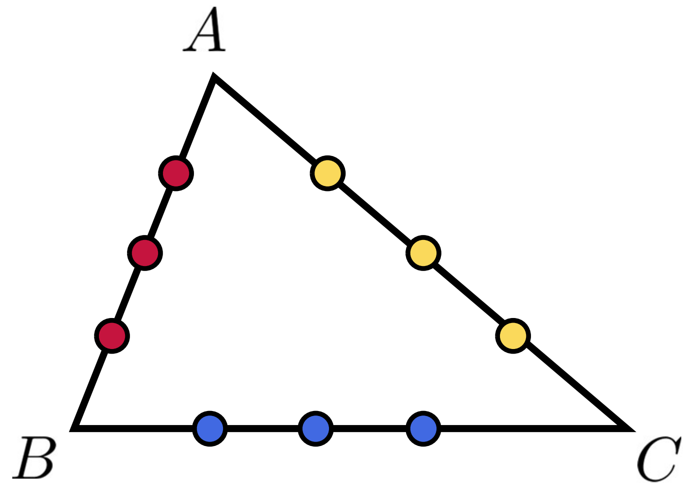
Counting Type 2

- Wall A: 6 choices
- Wall B: 5 choices
- Wall C: 1 choice
- Wall D: 5 choices

Number of type 2 $= 6 \times 5 \times 1 \times 5$

Q2: Colourful Triangles

In triangle ABC, there are 3 red points on side AB, 3 blue points on side BC and 3 yellow points on side AC. Soe Htet wants to connect three of these coloured points to create a (non-degenerate) triangle. In how many ways can he draw his triangle?

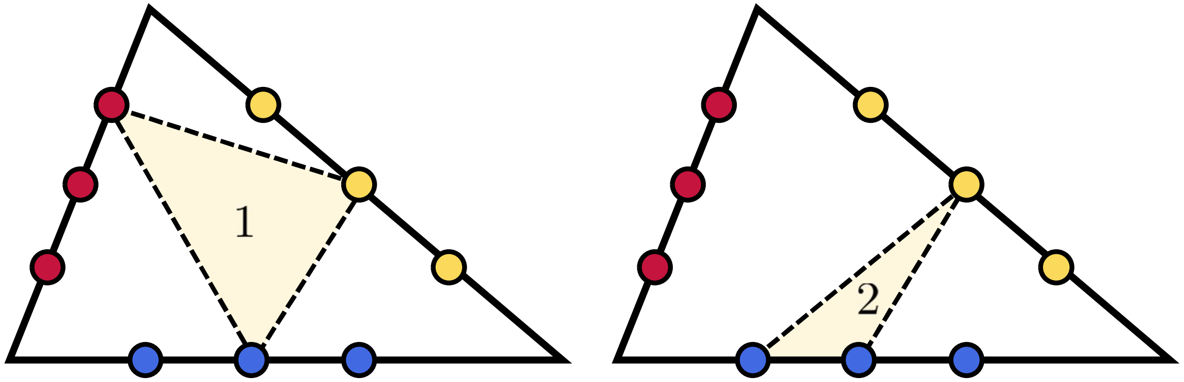


Answer: 81

Q2: Colourful Triangles

Solution: Group the triangles into two types:

- Type 1: Triangles with 3 colours,
- Type 2: Triangles with 2 colours.



Counting Type 1

- Decision 1: Choose a red point.
- Decision 2: Choose a blue point.
- Decision 3: Choose a yellow point.

Number of type 1 $= 3 \times 3 \times 3$

Counting Type 2

- Decision 1: Choose the colour for the base.
- Decision 2: Choose the base.
- Decision 3: Choose the tip.

Number of type 2 $= 3 \times 3 \times 6$

Section – II

When Cases are not Disjoint

Q3: Arranging HEAPS

Hnin Pwint wants to arrange the letters of the word HEAPS. But, since she really likes her name, she want the new word to start with a 'H', or end with a 'P' (or both).

For example, HAPSE, SHEAP, HASEP are good words for her.

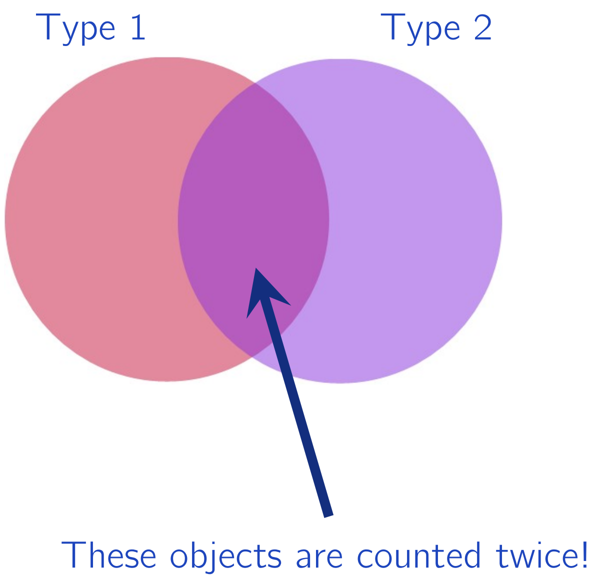
Answer: 42

Q3: Arranging HEAPS

Solution

We can make two types as usual.

- Type 1: Words starting with H, ← $4 \times 3 \times 2 \times 1 = 24$
- Type 2: Words ending with P. ← $4 \times 3 \times 2 \times 1 = 24$



If we add up, the words H _ _ _ P will be counted twice. So, we need to subtract the number of words H _ _ _ P.

↑

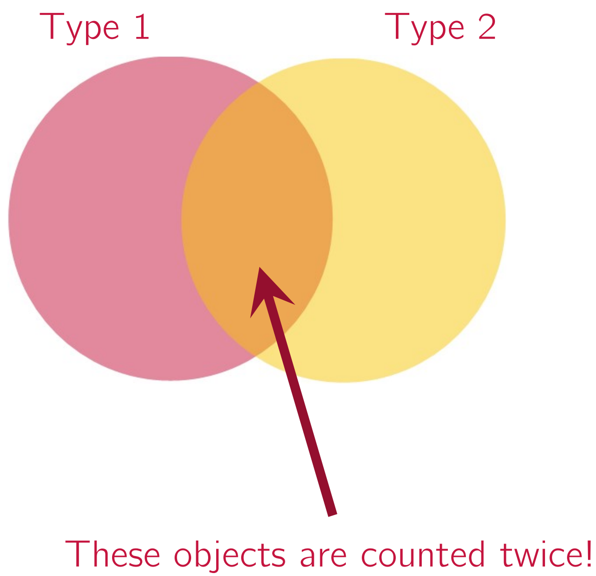
$3 \times 2 \times 1 = 6$

So, the answer is $24 + 24 - 6 = 42$.

Inclusion-Exclusion Principle (for 2 sets)

Suppose that we can group the given set of objects into two (possibly non-disjoint) groups. Suppose that

- there are N_1 objects in the 1st group,
- there are N_2 objects in the 2nd group,
- there are $N_{1,2}$ objects in both groups.



Then, the total number of objects is equal to $N_1 + N_2 - N_{1,2}$.

Question: What will be the formula for three groups?

Q4: Writing Multiples

Aye Mi and Bo Bo are writing down some numbers in their notebooks. Aye Mi writes down all multiples of 3 from 30 to 300. Bo Bo writes down all multiples of 5 from 50 to 500.

How many different numbers do they write down altogether?

Answer: 165

Q4: Writing Multiples

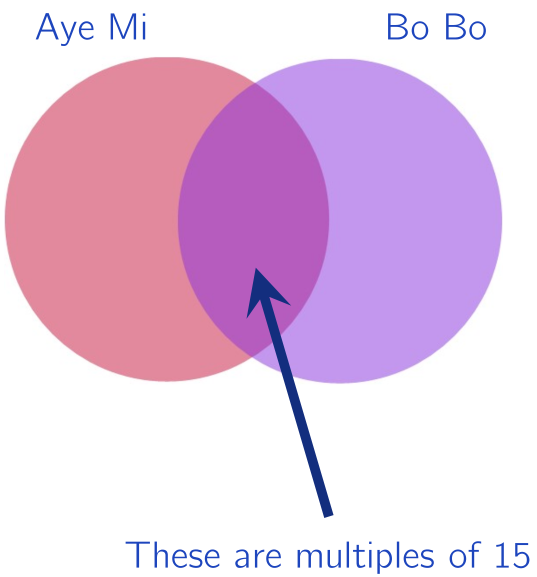
Solution

Aye Mi wrote down $\dfrac{300-30}{3}+1=91$ numbers.

Bo Bo wrote down $\dfrac{500-50}{5}+1=91$ numbers.

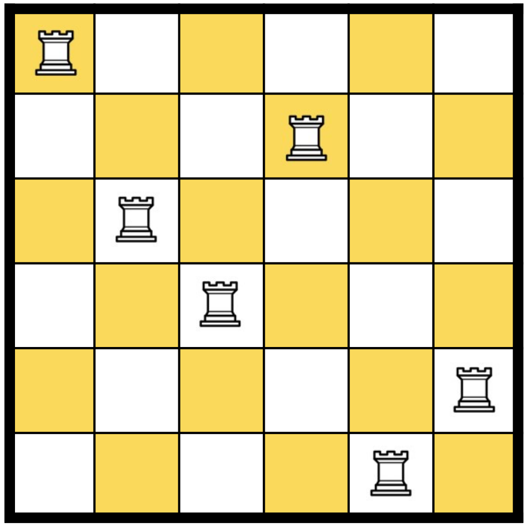
Both of them wrote down $\dfrac{300-60}{15}+1=17$ numbers.

So, altogether they wrote down $91+91-17=165$ different numbers.



Q5: Rooks on the Board

Six identical rooks are to be placed on a 6×6 chessboard. We want to place them so that no two rooks lie in the same row/column.



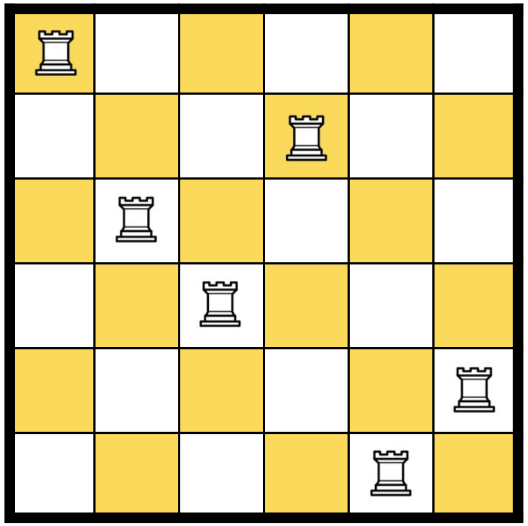
(a) In how many ways can we do this? Answer: 720

(b) In how many ways can we do this if we cannot place the rooks at the top-left and bottom-right corners? Answer: 504

Q5: Rooks on the Board

Solution

(a) This is just multiplication principle.



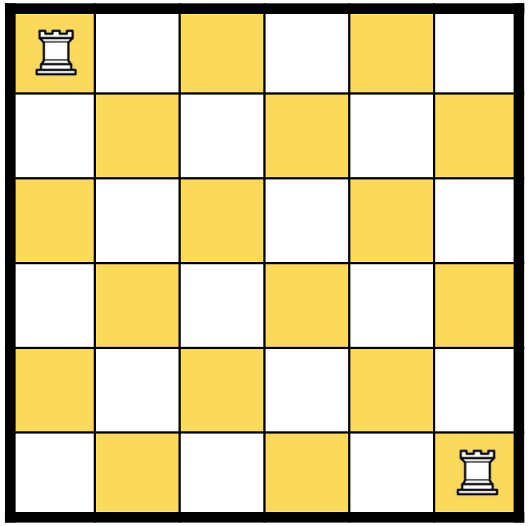
- Decision 1: Put a rook on the 1st row. ⟵ 6 choices
- Decision 2: Put a rook on the 2nd row. ⟵ 5 choices
- … ⟵ 4, 3, 2 choices
- Decision 6: Put a rook on the 6th row. ⟵ 1 choice

So, number of ways is $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

Q5: Rooks on the Board

Solution

Strategy: We will find what we don't want, and subtract from 720.



We can group the "bad" ways into two types

- Type 1: Those with a rook at top-left corner, ⟵ $5 \times 4 \times 3 \times 2 \times 1$
- Type 2: Those with a rook at bottom-right corner. ⟵ $5 \times 4 \times 3 \times 2 \times 1$

And, $4 \times 3 \times 2 \times 1 = 24$ ways are both type 1 and 2.

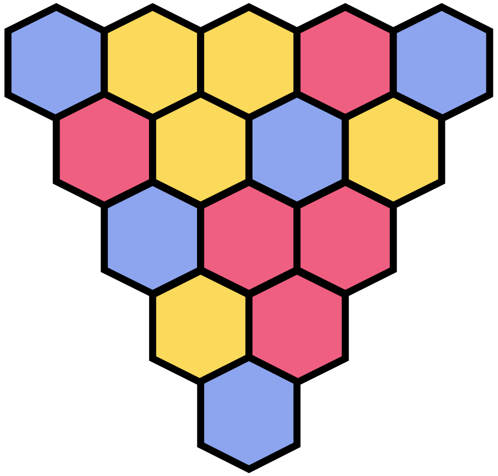
Thus, the answer is $720 - (120 + 120 - 24) = 504$

Section – III

Multiplication on Steroids

Q6: The Hexagon Triangle

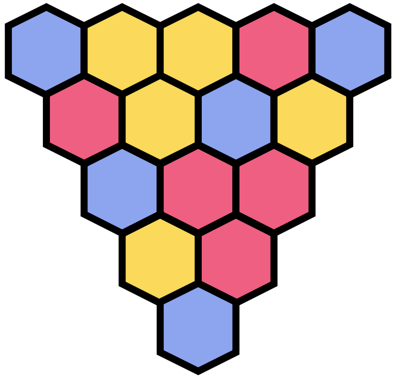
Consider a triangular grid of made up with 15 regular hexagons. We are to colour the interiors of the hexagons in either red, blue or yellow. Whenever three hexagons meet at a vertex, we either want them all to have the same colour, or to have all different colours. In how many ways can we do this?



Answer: 243

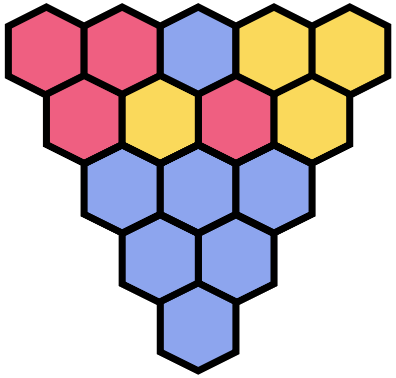
Q6: The Hexagon Triangle

Solution



Main Observations

- Colouring the first row is enough to fill out all hexagons.
- We can colour the first row whatever we want.



There are 3 choices for each hexagon in top row.

So, number of ways $= 3 \times 3 \times 3 \times 3 \times 3 = 243$.

Q7: Number of Divisors

How many divisors does the number 1200 have in total?

Answer: 30

Solution

First, note that $1200 = 2^4 \times 3 \times 5^2$.

So, each divisor = (some power of 2) × (some power of 3) × (some power of 5).

↑ 5 choices ↑ 2 choices ↑ 3 choices

Number of divisors is $5 \times 2 \times 3 = 30$.

Number of Divisors Formula

Let N be a positive integer with prime factorization: $2^{k_1} \times 3^{k_2} \times 5^{k_3} \times \cdots$.

Then, number of divisors of N is equal to $(k_1 + 1)(k_2 + 1)(k_3 + 1) \cdots$.

Examples

- Number of divisors of $2 \times 3 \times 7$ is equal to $(1 + 1)(1 + 1)(1 + 1) = 8$.
- Number of divisors of $2023 = 7 \times 17^2$ is equal to $(1 + 1)(2 + 1) = 6$.

Q8: Counting Subsets

Let n be a positive integer. How many subsets does the set $\{1, 2, 3, \ldots, n\}$ have?

Example

- For $n = 3$, the set $\{1, 2, 3\}$ has 8 subsets:

$$\begin{array}{cc} \emptyset & \{1,2\} \\ \{1\} & \{1,3\} \\ \{2\} & \{2,3\} \\ \{3\} & \{1,2,3\} \end{array}$$

Q8: Counting Subsets

Let's try for $n = 4$ first.

To make a subset of $\{1, 2, 3, 4\}$, we have to go through four decisions:

- Decision 1: To include or not to include 1.
- Decision 2: To include or not to include 2.
- Decision 3: To include or not to include 3.
- Decision 4: To include or not to include 4.

There are 2 choices for each decision.

So, number of subsets we can create is $2 \times 2 \times 2 \times 2 = 16$.

Q8: Counting Subsets

To make a subset of $\{1, 2, 3, \ldots, n\}$, we have to make n decisions:

- Decision 1: To include or not to include 1.
- Decision 2: To include or not to include 2.
- ...
- Decision n: To include or not to include n.

So, there are $\underbrace{2 \times 2 \times 2 \times \cdots \times 2}_{n \text{ factors}} = 2^n$ possible subsets.

That's it for this video.

Have a nice rest!