Lesson – 4

Repeated Permutations and Combinations

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Review

Combinations: Number of ways to select r objects out of n different objects is C_r^n .

Permutations: Number of ways to select and arrange r objects out of n different objects is P_r^n .

Formulae:

$$P_r^n = \underbrace{n \times (n-1) \times (n-2) \times \cdots}_{r \text{ terms}}$$

$$P_r^n = \frac{n!}{(n-r)!}$$

$$C_r^n = \frac{n \times (n-1) \times (n-2) \times \cdots}{r!}$$

$$C_r^n = \frac{n!}{r!(n-r)!}$$

Q1. Let's start repeating

Cho Thar wants to arrange the letters of the word AABBBCCCC.

In how many ways can she arrange them?

Solution

Prepare 9 blanks. Then, make 3 decisions.

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- Decision 1: Put A's in 2 out of 9 blanks. ⟵ C_2^9
- Decision 2: Put B's in 3 out of 7 remaining blanks. ⟵ C_3^7
- Decision 3: Put C's in 4 out of 4 remaining blanks. ⟵ C_4^4

Thus, number of ways is $C_2^9 \times C_3^7 \times C_4^4 = \frac{9 \times 8}{2!} \times \frac{7 \times 6 \times 5}{3!} \times \frac{4 \times 3 \times 2 \times 1}{4!} = \frac{9!}{2! \times 3! \times 4!}$

Q2. Let's repeat again

Cho Thar now wants to arrange the letters of the word ENGINEER.

In how many ways can she arrange them?

Solution

Prepare 8 blanks. Then, make 5 decisions.

- Decision 1: Put E's in 3 out of 8 blanks. ← C_3^8
- Decision 2: Put N's in 2 out of 5 remaining blanks. ← C_2^5
- Decision 3: Put G in 1 out of 3 remaining blanks. ← C_1^3
- Decision 4: Put I in 1 out of 2 remaining blanks. ← C_1^2
- Decision 5: Put R in 1 out of 1 remaining blank. ← C_1^1

E E E
N N
G
I
R

Thus, number of ways is $\frac{8\times 7\times 6}{3!}\times\frac{5\times 4}{2!}\times\frac{3}{1!}\times\frac{2}{1!}\times\frac{1}{1!}=\frac{8!}{3!\times 2!}$

Permutations with Repetitions

Suppose we have N objects. Out of these, n_1 are of one type, n_2 are of another type, ... , n_k are of the last type. Then, number of ways to arrange all N objects in a row is

$$\frac{N!}{n_1! \times n_2! \times \cdots \times n_k!}.$$

Reason: Exactly the same with Q1 and Q2 but written generally.

Q3. Repetition Practice

Write down the number of following objects in terms of factorials:

a. ways to arrange letters of the word ANIMATION, ← $\dfrac{9!}{2! \times 2! \times 2!}$

b. ways to arrange 3 blue marbles, 3 red marbles and 3 green marbles in a row (marbles of the same colour are identical), ← $\dfrac{9!}{3! \times 3! \times 3!}$

c. ways to shout "yes" 5 times and "no" 3 times, ← $\dfrac{8!}{5! \times 3!}$ ← C_5^8

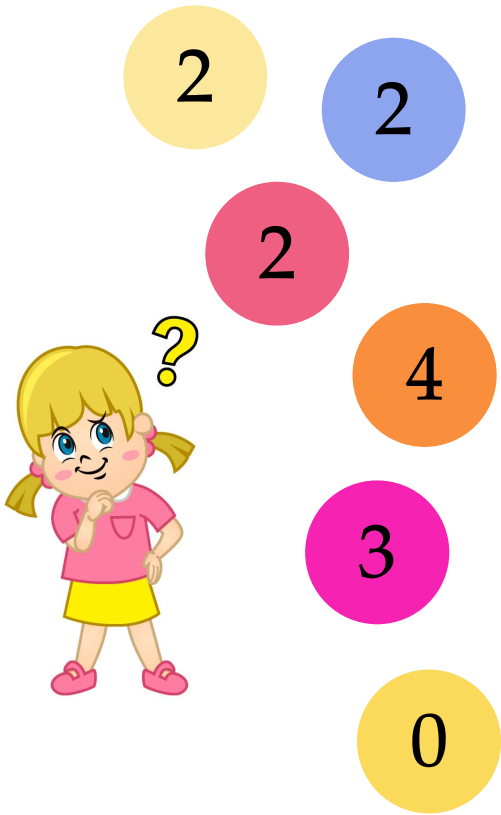
d. ways to seat 5 people in a row of 10 seats. ← $\dfrac{10!}{5!}$ ← P_5^{10}

Q4. Back to Digits

Hnin Pwint wants to arrange the digits of 242302 taken altogether to create a 6-digit number.

(a) How many numbers can she create?

(b) How many of those numbers are greater than 300000?



Q4. Back to Digits

Solution

(a) Number of ways to arrange 2, 2, 2, 3, 4, 0 is $\frac{6!}{3!} = 120.$

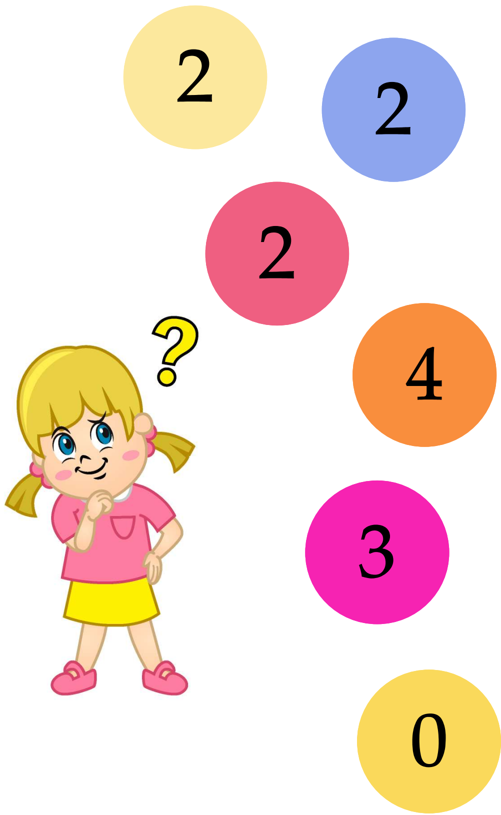
Out of these, $\frac{5!}{3!} = 20$ arrangements start with a 0.

So, number of "good" arrangements is $120 - 20 = 100.$

(b) Number of 6-digit numbers starting with 3 is $\frac{5!}{3!} = 20.$

Number of 6-digit numbers starting with 4 is $\frac{5!}{3!} = 20.$

So, the answer is $20 + 20 = 40.$



Q5. Blue Books Give Problems

Ten books are aligned in a row. 5 of them are green, 2 are yellow and 3 are blue. Books of the same colour are indistinguishable. In how many ways can we arrange the books if

(a) there are no restrictions?

(b) blue books must be together?

(c) no two blue books should be together?

Q5. Blue Books Give Problems

Solution

(b) Blue books must be together?

Put blue books in a box. Then, we have to arrange GGGGGYY[B].

So, number of ways is

$$\frac{8!}{5! \times 2!} = 168.$$

(c) no two blue books should be together?

Arrange GGGGGYY first. Then, place 3 B's in the 8 "gaps" formed.

So, number of ways is

$$\frac{7!}{5! \times 2!} \times C_3^8 = 1176.$$

Let's take a short break!

We will continue after 5 minutes.

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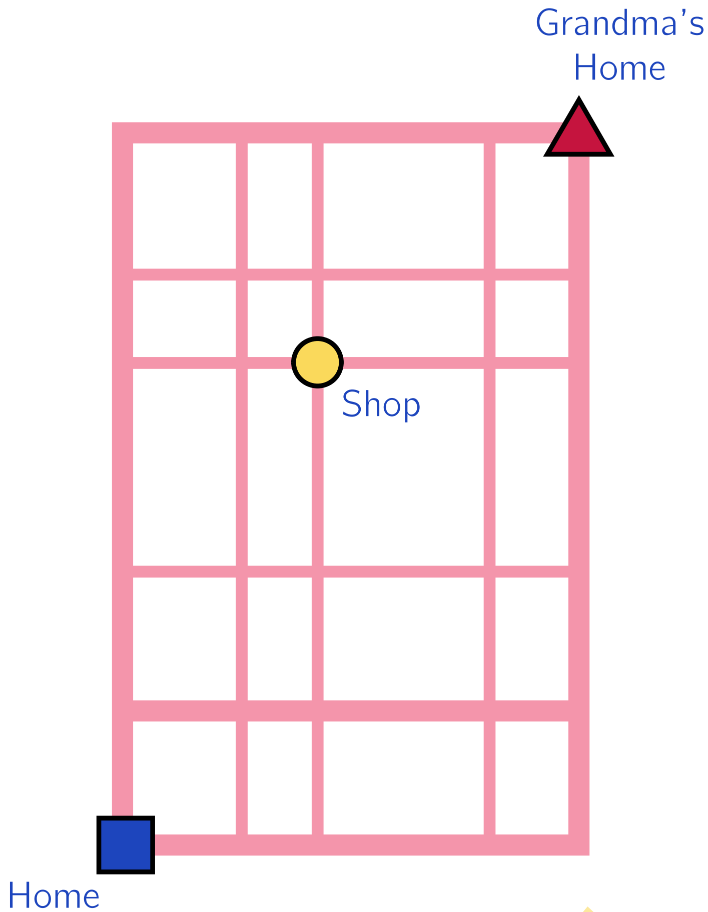
Q6. A Visit to Grandma

The given picture is the map of Kyaw Htet's town. Starting from his home, he wants to pay a visit to his grandma's home.

(a) How many shortest routes are there that he can take?

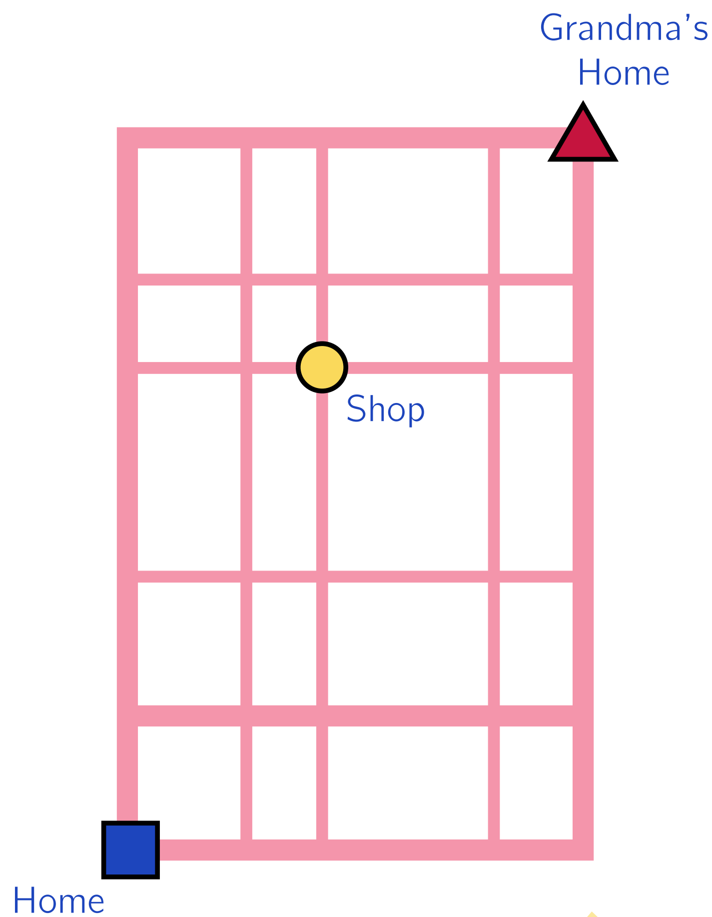
(b) How many shortest routes are there if he wants to go to the shop first to buy grandma's favourite snack?

Hint: Shortest routes are obtained by walking north and east only.



Q6. A Visit to Grandma

(a) How many shortest routes are there that he can take?



Solution

He needs to move east 4 times and north 5 times.

We can think of it as filling 9 blanks: 4 with east and 5 with north.

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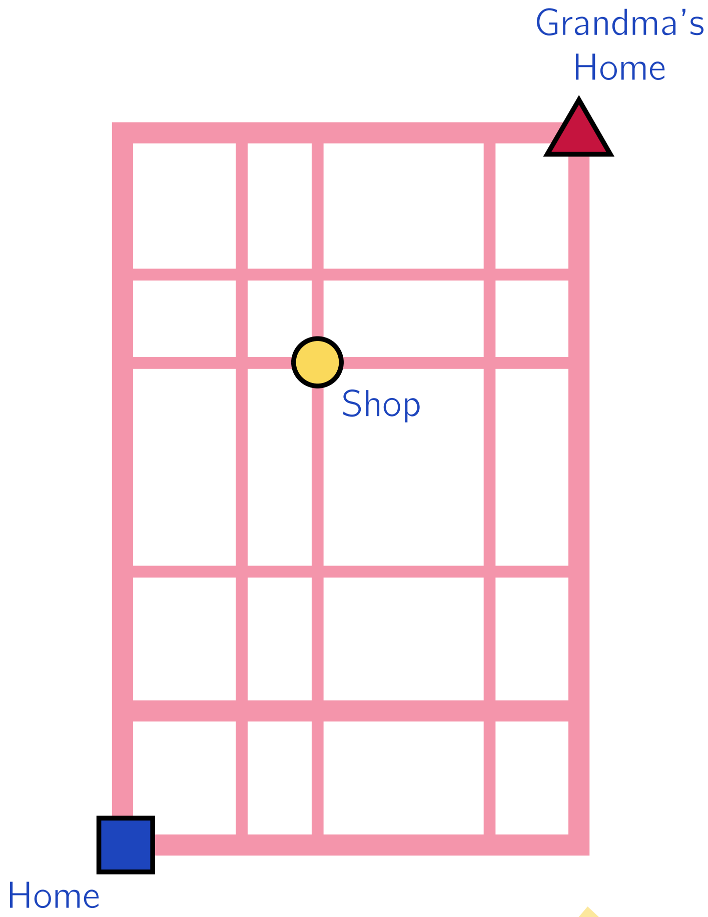
Therefore, the number of ways is $\dfrac{9!}{4! \times 5!} = 126.$

Q6. A Visit to Grandma

(b) How many shortest routes are there if he wants to go to the shop first to buy grandma's favourite snack?



Solution

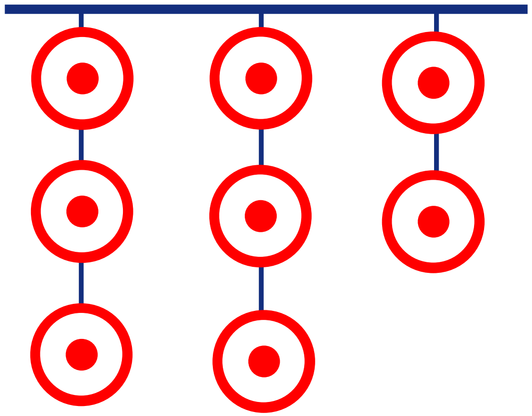
He needs to make 2 decsions:

- Decision 1: Go from home to shop, ← $\dfrac{5!}{2! \times 3!}$
- Decision 2: Go from shop to Grandma's house. ← $\dfrac{4!}{2! \times 2!}$

So, number of ways is $\dfrac{5!}{2! \times 3!} \times \dfrac{4!}{2! \times 2!} = 60.$

Q7. Target Practice

Eight targets are hung down from the ceiling as shown in the figure. The musketeer shoots down the targets, one at a time, but she does not shoot a target T unless all of the targets below T were shot. In how many ways can she shoot down all 8 targets?



Q7. Target Practice

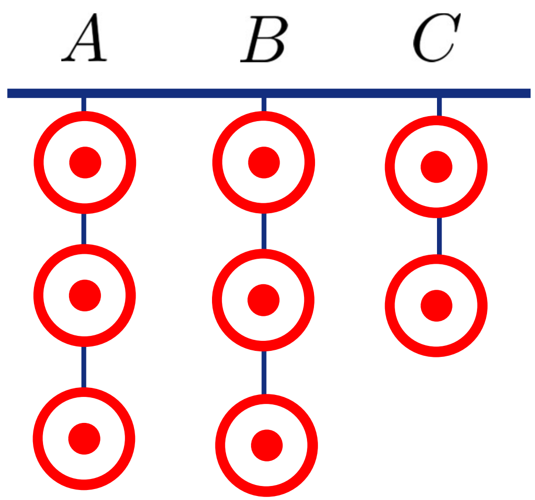
Solution

Label the columns as A, B and C.

Imagine if the shooter shouts the column as he shoots.

Then, he has to shout A 3 times, B 3 times and C 2 times.

Each shouting sequence gives a unique way to clear the targets.



Therefore, number of ways is equal to number of arrangements of AAABBBCC which is

$$\frac{8!}{3! \times 3! \times 2!} = 560.$$

Q8. Stars and Bars

Ko Thway has an infinite supply of red, blue and yellow marbles. Marbles of the same colour are indistinguishable. He wants to make a collection of 8 marbles. How many different collections can he create?

Q8. Stars and Bars

Solution

Imagine 8 stars arranged in a row: ★ ★ ★ ★ ★ ★ ★ ★

Ways to pick 8 marbles from red, blue and yellow can be represented with 8 stars and 2 bars and vice versa:

3 red, 3 blue, 2 yellow: ★ ★ ★ | ★ ★ ★ | ★ ★

6 red, 1 blue, 1 yellow: ★ ★ ★ ★ ★ ★ | ★ | ★

4 red, 4 blue: ★ ★ ★ ★ | ★ ★ ★ ★ |

5 red, 3 yellow: ★ ★ ★ ★ ★ || ★ ★ ★

8 blue: | ★ ★ ★ ★ ★ ★ ★ ★ |

Therefore, number of ways is $\dfrac{10!}{2! \times 8!} = 45.$

I guess we both earned our rest.

See you next week!

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