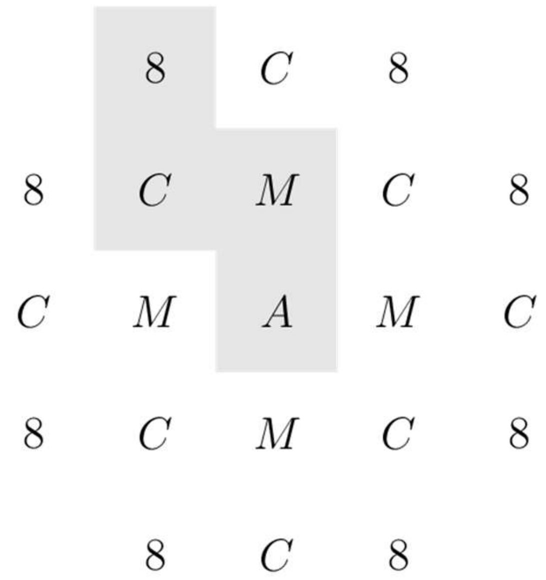
We will begin at 07:03 PM

Try this as a review in the mean time:

In the arrangement of letters and numerals as in the picture, by how many different paths can one spell AMC8? Beginning at the A in the middle, a path allows only moves from one letter to an adjacent (above, below, left, or right, but not diagonal) letter. One example of such a path is traced in the picture.



Solution to the Intro Problem

Solution

We have to do three decisions.

Decision 1: Choose which M to go. ← 4 ways

Decision 2: Choose which C to go. ← 3 ways

Decision 3: Choose which 8 to go. ← 2 ways

Therefore, number of possible ways is $4 \times 3 \times 2 = 24$.

	8	C	8	
8	C	M	C	8
C	M	A	M	C
8	C	M	C	8
	8	C	8	

Some Housekeeping

- Homework 2 is due tomorrow.
- The only way to learn the material is by working on questions by yourself. So, please make sure to put effort into homeworks. Do not care about the number of attempts.
- I may not be able to post the video tomorrow 07:00 PM. But I will make sure to post it on Wednesday 07:00 PM.

Lesson – 3

Combinations

COM

I

Selection versus Arrangement

What is the difference between these questions?

A small class has five students: Aye Aye, Bo Bo, Chaw Chaw, Dar Dar and Ei Ei.

Question 1: Sayarma wants to call 3 students to her desk. In how many ways can she do this? ⟵ We'll see in next slide.

Question 2: Sayarma wants to arrange 3 students in a line. In how many ways can she do this? ⟵ The answer is P_3^5

Answer: Question 1 is about selection, Question 2 is about selection and arrangement.

Resolving Question 1

Solution 1

The most straightforward way is to just write down all possible selections:

ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE

But this doesn't work if number of students is large.

Solution 2

Let X be the number of ways to call 3 students to the desk.

Then, $X \times P_3^3$ is the number of ways to arrange 3 out of 5 students in a row.

Therefore,

$$X \times P_3^3 = P_3^5 \longrightarrow X = \frac{P_3^5}{P_3^3} = \frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10.$$

Q1. Sayarma's Call

A class has 20 students. Sayarma wants to call 4 students to her desk. In how many ways can she do this selection?

Solution

Let X be the number of ways to do the selection.

Then, $X \times P_4^4$ is the number of ways to arrange 4 out of 20 students.

Therefore,

$$X \times P_4^4 = P_4^{20} \longrightarrow X = \frac{P_4^{20}}{P_4^4} = \frac{20 \times 19 \times 18 \times 17}{4 \times 3 \times 2 \times 1} = 4845.$$

Combinations

Number of ways to select r objects out of n different objects is denoted by C_r^n

For example,

- $C_3^5 = \dfrac{P_3^5}{P_3^3} = \dfrac{5 \times 4 \times 3}{3 \times 2 \times 1}$
- $C_4^{20} = \dfrac{P_4^{20}}{P_4^4} = \dfrac{20 \times 19 \times 18 \times 17}{4 \times 3 \times 2 \times 1}$

In general, we have the following formula:

$$C_r^n = \frac{P_r^n}{P_r^r} = \frac{n(n-1)(n-2)\cdots}{r!} = \frac{n!}{r!(n-r)!}$$

Comes from the fact that

$$P_r^n = \frac{n!}{(n-r)!}$$

Q2. Combinations Practice

The following numbers of ways can be written as C_r^n for some n and r.

- Ways to create 4-student group out of 10 students in the class. ⟵ C_4^{10}
- Ways to buy 3 out of 7 available fruits. ⟵ C_3^7
- Ways to draw a triangle using 8 given non-collinear vertices. ⟵ C_3^8
- Number of intersections in 6 non-parallel lines. ⟵ C_2^6

Choose-0 and Choose-1

Let n be a positive integer.

What is C_1^n and C_0^n?

Choose-1: $C_1^n = \frac{n}{1} = n.$
This also makes sense combinatorially.

Choose-0: $C_0^n = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \times n!} = 1.$

Choose-0 is a definition. It is quite difficult to make sense of it combinatorially.

Q3. Diagonals of a Decagon

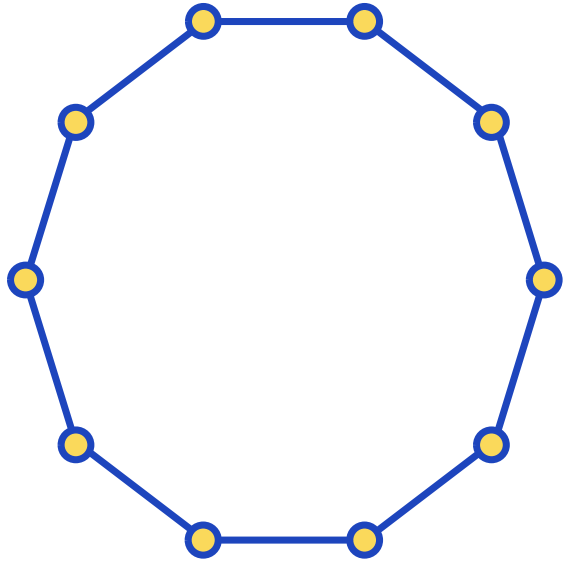
How many diagonals does a convex decagon have?

Solution

A diagonal is created selecting 2 non-adjacent vertices out of 10.

Therefore, number of diagonals is equal to

$$C_2^{10} - 10 = \frac{10 \times 9}{2} - 10 = 35.$$

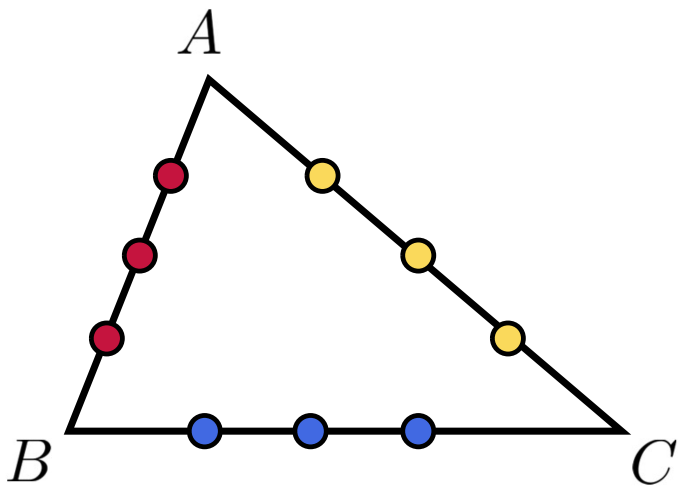


General Question: How many diagonals does a convex n-gon have?

Answer: $C_2^n - n = \frac{n(n-1)}{2} - n = n\left(\frac{n-1}{2} - 1\right) = \frac{n(n-3)}{2}.$

Q4. Colourful Triangles (again)

In triangle ABC, there are 3 red points on side AB, 3 blue points on side BC and 3 yellow points on side AC. Soe Htet wants to connect three of these coloured points to create a (non-degenerate) triangle. In how many ways can he draw his triangle?



Solution

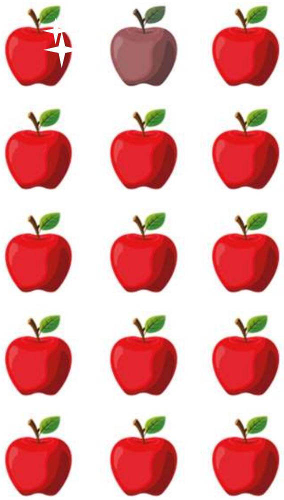
Number of ways to select 3 out of 9 points is C_3^9.

But, three of these selections do not create a triangle.

Therefore, number of ways to draw a triangle is $C_3^9 - 3 = 81$.

Q5. Steve's Apples

Steve went to a market to buy 4 apples. He found a seller offering 15 apples. One of them appears to be shiny and one of them is dull.



(a) In how many ways can he buy if he is not going to buy the dull apple?

(b) In how many ways can he buy if he is going to buy the shiny apple?

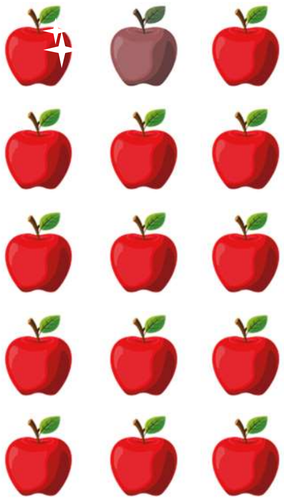
(c) In how many ways can he buy if he is going to buy the shiny one, but not to buy the dull one?

Q5. Steve's Apples

(a) In how many ways can he buy if he is not going to buy the dull apple?

Solution Throw the dull one in trash.

We need to select 4 out of 14 remaining apples. So, the answer is C_4^{14}.



(b) In how many ways can he buy if he is going to buy the shiny apple?

Solution Put the shiny in your basket.

We need to select 3 out of 14 remaining apples. So, the answer is C_3^{14}.

(c) In how many ways can he buy if he is going to buy the shiny one, but not to buy the dull one?

Solution Throw the dull one in trash and put the shiny in basket.

We need to select 3 out of 13 remaining apples. So, the answer is C_3^{13}.

Let's have a break!

We'll continue in 5 minutes.

COM

I

Q6. Consonants and Vowels

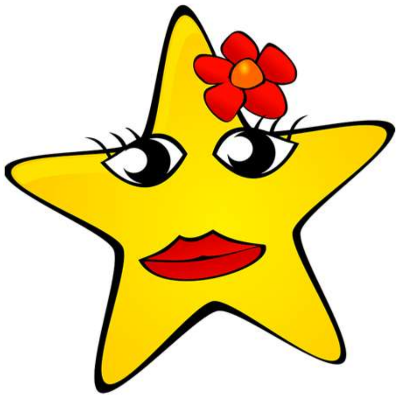
Kyal Kyal wants to create a 5-letter word using the letters A-Z each no more than once. How many words with 3 vowels and 2 consonants can she create?

(Fact: There are 5 vowels and 21 consonants from A-Z)

Solution

She has to do three decisions:

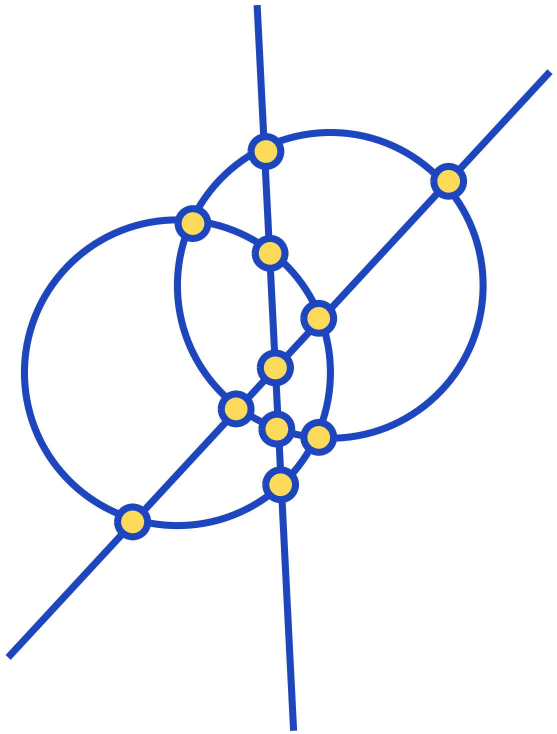
- Decision 1: Select 3 vowels out of 5. ← C_3^5
- Decision 2: Select 2 consonants out of 21. ← C_2^{21}
- Decision 3: Make a 5-letter word with chosen letters. ← P_5^5



So, the answer is $C_3^5 \times C_2^{21} \times P_5^5 = 252000.$

Q7. Circles and Lines

Three circles and four lines are going to be drawn on the plane. What is the maximum number of intersections can we obtain?

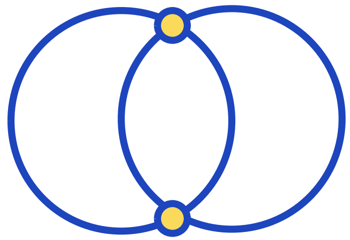


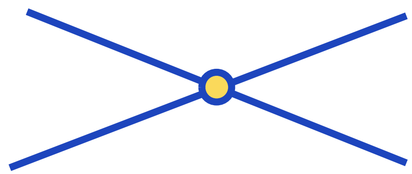
Q7. Circles and Lines

Solution

To maximize the intersections,

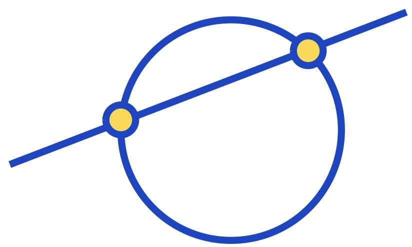
- Any two circles should intersect at 2 points
- Any two lines should intersect at a point
- Any circle and line should intersect at 2 points.
- No three objects (lines or circles) shall meet at a point.





So, we will count 3 types of intersections:

- Type 1: Circle and circle ← $2 \times C_2^3$
- Type 2: Line and line ← C_2^4
- Type 3: Circle and line ← $2 \times 3 \times 4$



The answer is $2 \times C_2^3 + C_2^4 + 2 \times 3 \times 4 = 6 + 6 + 24 = 36.$

Q8. Adventurer Pack

A pack of 5 adventurers is going to be created from 6 dwarfs and 7 elves. In how many ways can we create the pack if it has to contain at least one dwarf and at least one elf?

Q8. Adventurer Pack

Solution

The pack has 6 types based on its member-composition:

- Type 1: No dwarfs, 5 elves ← C_5^7
- Type 2: 1 dwarf, 4 elves ← $C_1^6 \times C_4^7$
- Type 3: 2 dwarfs, 3 elves ← $C_2^6 \times C_3^7$
- Type 4: 3 dwarfs, 2 elves ← $C_3^6 \times C_2^7$
- Type 5: 4 dwarfs, 1 elves ← $C_4^6 \times C_1^7$
- Type 6: 5 dwarfs, no elves ← C_5^6

We want types 2, 3, 4, 5. Just add those up to get the answer.

Q8. Adventurer Pack

Alternate Solution

The pack has 6 types based on its member-composition:

- Type 1: No dwarfs, 5 elves ← C_5^7
- Type 2: 1 dwarf, 4 elves ← $C_1^6 \times C_4^7$
- Type 3: 2 dwarfs, 3 elves ← $C_2^6 \times C_3^7$
- Type 4: 3 dwarfs, 2 elves ← $C_3^6 \times C_2^7$
- Type 5: 4 dwarfs, 1 elves ← $C_4^6 \times C_1^7$
- Type 6: 5 dwarfs, no elves ← C_5^6

We don't want types 1 and 6. So, we just need to subtract them from total.

Thus, the answer is $C_5^{13} - C_5^7 - C_5^6$.

I guess we both earned our rest.

See you soon!

COM

I