Construction of AP-free Sets in $\mathbb{N}$

Presented by Hein Thant AUNG

21 March 2023

For each positive integer n, let $r_3(n)$ be the size of the largest subset A of $\{0, 1, \dots, n\} := [n]$ that does not contain a non-trivial 3-term arithmetic progression. We will call such sets *AP-free*. Last time, we mentioned (and proved as a bonus corollary of Triangle Removal Lemma) Roth's theorem [2] which concerns the upper bound of $r_3(n)$. This time, we will discuss some attempts to lower bound $r_3(n)$ and describe one possible way to lift these constructions into abelian groups with no element of order 2.

1 Greedy Construction

The most straightforward way to construct an AP-free set $A \subseteq [n]$ is to construct it in a greedy manner:

- start with $A_0 = \{0\}$,
- After constructing A_k, find the smallest positive integer α_k in $[n]$ (if one exists) such that $A_k \cup \{\alpha_k\}$ is AP-free and define $A_{k+1} = A_k \cup \{\alpha_k\}$. Define $A = A_k$ if such α_k cannot be found.

We will show that this construction gives $|A| \leq n^{\log 2/\log 3}$ which follows immediately from the following curious observation:

Theorem 1. Let $A \subseteq [n]$ be the AP-free set constructed greedily as in the above discussion. Then,

$$A = \{m \in [n] : \text{ base-3 representation of } m \text{ does not contain the digit } 2\}.$$

Proof. We induct on k i.e. suppose that all elements of A_k does not contain the digit 2 in its base-3 representation and that it contains all such numbers less than α_k. Suppose to the contrary that base-3 representation of α_k contains a 2. Then, let β be obtained by changing all 2s into 1s. Then, $\beta < \alpha_k$ so that $\beta \in A_k$ and $\alpha_k + \beta$ has all 2s in its base-3 representation. Hence, $(\alpha_k + \beta)/2$ is an integer in A_k, contradiction. □

2 Behrend's Construction

Many constructions better than greedy approach have already been found years ago; the construction we are going to give today is due to Behrend [1]:

$$r_3(n) \geq n^{1-\frac{2\sqrt{2\log 2}+\varepsilon}{\sqrt{n}}}$$

for sufficiently large $n(\varepsilon)$ for any $\varepsilon > 0$. In particular, this shows that $r_3(n) \geq n^{1-\varepsilon}$ for any $\varepsilon > 0$. The general idea of Behrend's proof is as follows:

- For some positive integers N and d, consider a collection $S \subseteq [N]^d$ that lie on some sphere,
- Since spheres do not contain collinear points, S does not contain arithmetic progressions as a subgroup of $\mathbb{Z}^d$,
- Embed $[N]^d$ into $[n]$ while preserving the additive structure of $[N]^d$.

We begin by precisely defining what it means to embed some set inside another while preserving the additive structure.

Definition 2. Let G_1 and G_2 be abelian groups and take $A \subseteq G_1$ and $B \subseteq G_2$. For a positive integer k, a function $\varphi : A \to B$ is called a *Freiman k-homomorphism* if whenever $x_1 + \cdots + x_k = y_1 + \cdots + y_k$ where $x_i, y_i \in A$, we have

$$\varphi(x_1) + \cdots + \varphi(x_k) = \varphi(y_1) + \cdots + \varphi(y_k).$$

If φ is a bijection with φ^{-1} also a Frieman k-homomorphism, then we say φ is a Frimen k-isomorphism.

Theorem 3. Let $A = [N-1]^d \subseteq \mathbb{Z}^d$ and define $\varphi : A \to \mathbb{Z}$ by

$$\varphi(a_0, \ldots, a_{d-1}) = a_0 + a_1(kN-1) + a_2(kN-1)^2 + \cdots + a_{d-1}(kN-1)^{d-1}.$$

Then, φ is a Frieman k-isomorphism with between A and $\varphi(A)$.

Proof. Since every integer in $\mathbb{Z}$ has a unique representation in base-$(kN-1)$, it follows that φ is injective and well-defined. Now, consider $\mathbf{x}_1, \ldots, \mathbf{x}_k$ and write

$$\varphi(\mathbf{x}_i) = x_{i0} + x_{i1}(kN-1) + x_{i2}(kN-1)^2 + \cdots + a_{i(d-1)}(kN-1)^{d-1}$$

for each $i = 1, \ldots, k$. Then,

$$\varphi(\mathbf{x}_1)+\cdots+\varphi(\mathbf{x}_k) = (x_{10}+\cdots+x_{k0})+(x_{11}+\cdots+x_{k1})(kN-1)+\cdots+(x_{1(d-1)}+\cdots+x_{k(d-1)})(kN-1)^{d-1}. \tag{2.1}$$

Note that $x_{1j} + \cdots + x_{kj} < (N-1) + N + N + \cdots + N = kN - 1$ and hence addition (up to k terms) is done digit-wise. It follows that both φ and φ^{-1} are Frieman k-homomorphisms. □

Corollary 4. A subset A of $[N-1]^d$ is AP-free in $\mathbb{Z}^d$ if and only if $\varphi(A)$ is AP-free in $[(2N-1)^d]$.

We may now begin to present the proof of Behrend in detail. For a given positive integer n and d, let N be a positive integer such that

$$(2N-1)^d \le n < (2N+1)^d.$$

Now, partition $[N-1]^d$ into spheres:

$$S_r = \{\mathbf{x} \in [N-1]^d : \|\mathbf{x}\|_2 = r\}$$

such that $[N-1]^d = S_{\sqrt{0}} \sqcup S_{\sqrt{1}} \sqcup \cdots \sqcup S_{\sqrt{d}(N-1)}$. Therefore, by pigeonhole principle, there exists some sphere S_r such that

$$|S_r| \geq \frac{N^d}{1+d(N-1)^2} > \frac{N^{d-2}}{d}.$$

Since S_r is AP-free in $[N-1]^d$, its image A under the Freiman 2-homomorphism is also AP-free in $[(2N-1)^d]$. Now, we have

$$r_3(n) \geq r_3((2N-1)^d) > \frac{N^{d-2}}{d} > \frac{(n^{1/d}-1)^{d-2}}{d2^{d-2}} = \frac{n^{1-(2/d)}}{d2^{d-1}} \cdot 2\left(1-\frac{1}{n^{1/d}}\right)^{d-2}.$$

Note that the factor $2(1-n^{-1/d})^{d-2}$ is greater than 1 for large enough n and for a "well-chosen " funciton d of n. It turns out that the best d we can choose is

$$d = \left[\sqrt{\frac{2\log n}{\log 2}}\right]$$

where $[x]$ is the floor function. Under this choice, for sufficiently large n, we have

$$r_3(n) > \frac{n^{1-(2/d)}}{d2^{d-1}} = n^{1-\frac{2}{d}-\frac{\log d}{\log n}-\frac{(n-1)\log 2}{\log n}} > n^{1-\frac{2\sqrt{2\log 2}+\varepsilon}{\sqrt{n}}}$$

for every $\varepsilon > 0$.

References

[1] Felix A Behrend. "On sets of integers which contain no three terms in arithmetical progression". In: *Proceedings of the National Academy of Sciences* 32.12 (1946), pp. 331–332.

[2] Klaus F Roth. "On certain sets of integers". In: *J. London Math. Soc* 28.1 (1953), pp. 104–109.